Time dependent solutions of the Navier-Stokes and Euler equations and Turbulence.

Claude Bardos

Retired, Laboratoire Jacques Louis Lions, Université Pierre et Marie

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Introduction

The goal of this talk is to show what type of progress (if any!) the mathematical analysis of deterministic phenomena (rather than statistical theory of turbulence) can bring to the understanding of turbulence.
Emphasis is put on the Euler equation and on the large Reynolds number limit.
Beside my personal taste and choice two good reasons for that :

- For the Navier-Stokes equation, technical (but not easy at all) progress have been obtained since the work of Leray mostly contributing to the understanding of the validity of the equation as a model.
- For the Euler quation many results have been recently produced with a rather accessible interpretation.

Introduction

- **Consider , Discuss, Use, Weak solutions, Dissipative solutions of Euler equations.**

- **Discuss the relation between boundary effects and generation of turbulence.**

Introduction

Analyzing the turbulence generated by boundary effects, one finds a series of equivalent criteria for the absence of turbulence which would define turbulence as a situation where any of this effects is present:
More precisely there is no turbulence if one of the following effect is present:

- No anomalous dissipation of energy.
- No non trivial Reynolds stress tensor.
- No production of the vorticity at the boundary.
- No production of vorticity at a boundary layer of size ν
- No detachement .

If turbulence is present none of the above statement is true. The Prandlt equation of the boundary layer are not valid. There is a non trivial Reynolds tensor which is the spectra in the sense of Kolmogorov-Heisen -berg and which in the mathematical sense is a non trivial Wigner Measure.

Contents

As a consequence the talk is organized as follow:

- Presentation of the Navier-Stokes equation with emphasis on the Reynolds Number.
- Comparison between the notion of weak convergence and the issues of statistical theory.
- Cauchy problem for the smooth solutions of the Euler equation (Cauchy Problem is a name for stability properties).
- Notions of weak solutions.
- The wilde weak solutions of Scheffer, Shnirleman, De Lellis and Skezehelydi.

Contents

- The notion of viscosity solutions and dissipative solutions.
- The relation between convergence to the Euler equation and the non appearance of turbulence as it was formulated by T. Kato.
- Some comments on the situation for the planar flows ($2d$ Euler equation).
- Justification of the point of view of Kato by doing a similar analysis at the level of the convergence of solutions of Boltzmann equations to solutions of the Euler equation.
- Conclusion.

The Navier-Stokes Equations

$$\partial_t u_\nu + u_\nu \cdot \nabla u_\nu - \mu^* \Delta u_\nu + \nabla p_\nu = 0$$
$$\nabla \cdot u_\nu = \sum_{1\le i\le d} \partial_{x_i} (u_\nu)_i = 0 \,, \; u_\nu \cdot \nabla u_\nu = \sum_{1\le i\le d} (u_\nu)_i \partial_{x_i} u_\nu \,.$$

Called incompressible because of the relation $\nabla \cdot u = 0\,.$
But also equations for fluctuations: ϵ the Mach number ratio between the fluctuation of velocity and the sound speed:

$$u = \epsilon \tilde{u} \quad \theta = 1 + \epsilon \tilde{\theta}\,, \rho = 1 + \epsilon \widetilde{\rho}$$
$$\nabla_x \cdot \tilde{u} = 0\,, \quad \partial_t \tilde{u} + \tilde{u} \cdot \nabla_x \tilde{u} + \nabla_x \tilde{p} = \mu^* \Delta \tilde{u}\,,$$

The Navier-Stokes Equations

Density and temperature fluctuations $\widetilde{\rho}, \tilde{\theta}$ passive scalars:

$$\widetilde{\rho} + \tilde{\theta} = 0\,, \qquad \text{Boussinesq approximation}$$
$$\tfrac{d+2}{2}(\partial_t \tilde{\theta} + \tilde{u}\cdot\nabla_x \tilde{\theta}) = \kappa^* \Delta\tilde{\theta} \qquad \text{Fourier Law.}$$

Phenomenological derivation or consequence of the Boltzmann equation Hilbert 6th problem.

In general ν in Navier-Stokes not the real viscosity of the fluid but the inverse of the Reynolds number, a rescaled viscosity adapted to the size of the fluctuations of the velocity is given by the formula:

$$\mathcal{R}e = \frac{UL}{\mu^*}$$

In all practical applications $\mathcal{R}$ is very large therefore ν is very small. Bicycle 10^2, Industrial fluids (pipes ships...) 10^4, Wings of airplanes 10^6, Space Shuttle10^8, Weather Forcast, Oceanography 10^{10}, Astrophysic 10^{12}.

It would be natural to study the limit $\nu \to 0$ in the Navier-Stokes equation or even to put $\nu = 0$ and then consider the Euler equation... Things are not so simple but may be useful. Almost no basic mathematical progress has been done on the understanding of the structure of solutions of the Navier-Stokes equations since Leray.. On the other hand important progress have been done concerning the Euler Equation...May be because it is more "mathematical. "

Connect these progress with the law of Kolmogorov for turbulence

With convenient boundary conditions:

$$\frac{d}{dt}\int_\Omega \frac{|u(x,t)|^2}{2} + \nu\int_\Omega |\nabla u(x,t)|^2 dx \le 0\,,$$

$$< .,. > \text{ statistical average} \simeq\simeq \overline{u_\nu} \text{ weak limit}\,,$$

$$(\langle u_\nu \otimes u_\nu\rangle - \langle u_\nu\rangle\langle u_\nu\rangle) \quad \text{Reynolds stress tensor}\,,$$

$$0 \le \lim_{\nu\to 0}(u_\nu - \overline{u_\nu})\otimes(u_\nu - \overline{u_\nu}) = \lim_{\nu\to 0}(u_\nu\otimes u_\nu - \overline{u_\nu}\otimes\overline{u_\nu}) \text{ w-Reynolds s.t.}$$

$$0 < \epsilon = \nu\langle|\nabla u_\nu|^2\rangle \simeq \frac{\nu}{T}\int_0^T\int_\Omega |\nabla u_\nu|^2 dxdt \text{ Kolmogorov hypothesis}\,,$$

$$\langle|u(x+r) - u(x)|^2\rangle^{\frac12} \simeq (\nu\langle|\nabla u|^2\rangle)^{\frac23}|r|^{\frac13}\text{Kolmogorov law}\,.$$

- Law is in average with a forcing term.
- Boundary effects are physical forcing terms.
- The averaged energy dissipation $(\nu\langle|\nabla u_\nu|^2\rangle)^{\frac{2}{3}}$ does not go to zero! as $\nu \to 0$.
- The averaged regularity is $\frac{u_\nu(x+r)-u_\nu(x)|}{r^{\frac{1}{3}}}$ bounded in Hölder space $C^{\frac{1}{3}}$

Basic estimates for 3d Euler equation energy and vorticity

$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0\,,\ \nabla \cdot u = 0\,,\ \text{in } \Omega\,.$$

$$u \cdot \vec{n} = 0 \ \text{ on } \partial\Omega\,,$$

$$\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u \text{ in } \Omega$$

$$\nabla \cdot u = 0\,, \nabla \wedge u = \omega \text{ in } \Omega\,,\ \text{and } u \cdot \vec{n} = 0 \ \text{ on } \partial\Omega$$

Formal Energy conservation $\frac{d}{dt}\int_\Omega \frac{|u(x,t)|^2}{2} dx = 0$

Formal 2d Vorticity transport $\partial_t \omega + u \cdot \nabla \omega = 0\,,$

$$\dot{x}(s) = u(x(s), s)\,, \frac{D}{Dt}\omega = \frac{d}{dt}\omega(x(t), t) = 0.$$

$$\partial_t u_\nu + \nabla \cdot (u_\nu \otimes u_\nu) - \nu \Delta u_\nu + \nabla p = 0\,, u_\nu = 0 \text{ on } \partial\Omega$$

$\Rightarrow \frac{1}{2}\frac{d}{dt}\int_\Omega |u_\nu(x,t)|^2 dx + \nu \int_\Omega |\nabla u_\nu(x,t)|^2 dx = 0$ Formal energy cons.

One word about Cauchy Problem for the Euler equation

$$\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u \Leftrightarrow \frac{d\omega}{dt} = \omega \nabla K(\omega) \simeq \omega^2$$

Compared with

$$y' = Cy^2 \,, \qquad y(t) = \frac{1}{1 - Cty(0)} \,.$$

• Local existence, stability of smooth solution in good mathematical spaces $C^{1,\alpha}$ and persistence of higher stability results.

• Instability of solutions in less regular spaces. Example of the shear flow

$$u(x,t) = (u_1(x_2), 0, u_3(x_2, x_1 - tu_1(x_2))) \quad \partial_{x_2} u_3 = \partial_2 u_3 - t\partial_3 u_3 \partial_1 u_1$$

$u(0,x) \in C^{0,\alpha}$ implies $u(t,x) \in C^{0,\alpha^2}$ not imply $u \in C^{0,\alpha}$. for $t > 0$.

• With smooth initial data loss of regularity is an open problem driven by $\sup |\omega(x,t)|$ (Beale Kato Majda) or absence of oscillations in the direction of vorticity (Constantin Fefferman Majda). But at least existence, uniqueness and stability are assumed for short time.

Weak solutions

Why weak solutions?

- Naturally in $L^2(\Omega)$ (energy controlled)!
- Weak solution are compulsory in compressible Euler equation with the presence of shocks.
- Since global existence of smooth solution is an open problem why not to consider weak solution.
- Weak solutions are compatible with weak convergence from Navier-Stokes to Euler.
- The main difficulty is the limit (whatever construction would be proposed) in the nonlinear term. For weak limit : subsequences bounded in $L^\infty(\mathbb{R}_t; L^2)$ the w-Reynolds S.T. may not be 0 .
- Weak solutions are the only objects compatible with anomalous dissipation of energy:

$$\int_\Omega \nabla \cdot (u \otimes u) u dx \neq 0 \text{ or even not defined}$$

Weak solutions, Definition

$$\phi \in C^\infty(\mathbb{R}_t \times \mathbb{R}^d_x)\text{With compact support} \ ,$$

$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0, \nabla \cdot u = 0 \quad u \cdot \vec{n} = 0 \text{ on } \partial\Omega \,,$$

$$\Leftrightarrow$$

$$\int_{\mathbb{R}_t \times \Omega} (u\partial_t \phi + (uS(\phi)u) + p\nabla \cdot \phi) dxdt = 0 \,,$$

$$S(\phi) = \frac{\nabla\phi + \nabla^t\phi}{2} \quad \text{and } u(x,0) = u_0(x) \,.$$

Onsager criteria for conservation of energy for weak solutions:

$$0 = \int_0^T \int_\Omega \nabla(u \otimes u) u dx dt \simeq \int_0^T \int_\Omega |\nabla^{\frac{1}{3}} u|^3 dx dt \simeq (\nabla^{\frac{1}{3}} u) \in L^3(\Omega \times 0, T)$$

Proof by Constantin, E and Titi in $L^3(\mathcal{B}^{\frac{1}{3}}_{3,\infty})$
Conversely conservation of energy for weak solution does not implies regularity. Simple example with the shear flow.

Weak-Strong Stabilty

u weak solution and w smooth solution (with $\nabla w \in L^1(0,T;L^\infty(\Omega))$) then one should have local stability:

$$\int_\Omega |u(x,t)-w(x,t)|^2 dx \le (\int_\Omega |u(x,0)-w(x,0)|^2 dx) e^{2\int_0^t |S(w(s))|_\infty ds}$$

Theorem

Any weak solution which satisfies the relation:

$$E(t) = \frac{1}{2}\int_\Omega |u(x,t)|^2 dx \le \frac{1}{2}\int_\Omega |u(x,0)|^2 dx = E(0)$$

satisfies the weak strong stability estimate (in particular local in time uniqueness for smooth initial data).

These statement have to be compared with a family of "mathematical results " Scheffer, Shnirelman, De Lellis, Szekelyhidi and Wiedeman.

Theorem

- For any initial data (even a regular one) $u_0 \in L^2(\Omega)$ there exists an infinite family of global in time weak solutions.
- Given any domain $\tilde{\Omega}$ with $\overline{\tilde{\Omega}} \subset\subset \Omega$ and any positive function $e(x,t) \geq 0$ with the following properties

$$\text{support } e(x,t) \in C^\infty(\overline{\tilde{\Omega}} \times [-T,T])$$
$$x \in \overline{\tilde{\Omega}} \times [-T,T] \Rightarrow e(x,t) > 0\,,$$
$$\text{and } x \notin \overline{\tilde{\Omega}} \times [-T,T] \Rightarrow e(x,t) = 0\,.$$

Then there exists and infinite set of data $u(x,0)$ with weak solutions such that:

$$u(.,t) \in C(]-T,T[, L^2(\Omega))\,, \quad u(.,0) = u_0\,,$$
$$\frac{1}{2}|u(x,t)|^2 = e(x,t) \Rightarrow E(t) = \frac{1}{2}\int |u(x,t)|^2 dx = \int e(x,t)dx \ \text{ prescribed}$$

In particular, there exists an infinite set of initial data with an infinite number of solutions of the Cauchy problem.

Observe that these solutions are compact in space time and existence of such objects in the real world would solve the energy crisis. This seems also to indicate that the Euler equations by themselves are not completely physical...With the d'Alembert paradox the validity of such equations would also imply that the planes do not fly.

Figure: Euler, D'Alembert, Navier and Stokes

⇒Importance of Viscosity and Boundary conditions.

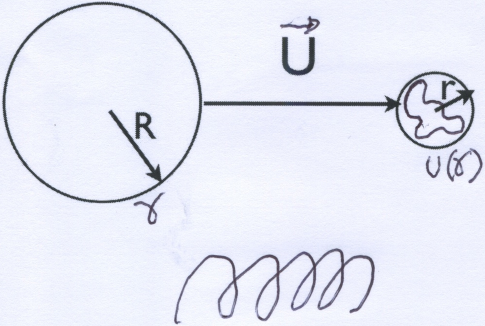
- Observe that if $u(x,t)$ is regular on $\Omega\times]0,T[$ any weak solution $w(x,t)$ such that $w(x,0)=u(x,0)$ must be of increasing energy for $t>0$.
- Keeping in mind that $1/3$ type regularity implies conservation of energy and the existence of solution $u\in C(\mathbb{R}_t;L^2(\Omega))$, such that the energy is not conserved leads to the following questions:

1 Does there exist a threshold α such that regularity greater than α guarantee the conservation of energy

2 Does there exists solutions just less regular than α for which energy is not conserved. At present one has energy conservation with $\alpha>\frac{1}{3}$ and existence of solutions with $u\in C(\mathbb{R}_t,L^2)$ more or less $\alpha=0$, which means a huge gap... *Shortening the gap is an open problem.*

Some comments about the construction of the wild solutions

Present proof shares much similarities with the problem of the invariant imbedding:
Both use the formalism of differential inclusions and accumulation of oscillations.
The problem of isometric imbedding in $\mathbb{R}^2$: Is it possible for any $r > 0$ to construct a map $U = (u_1(x_1, x_2), u_2(x_1, x_2))$ $\mathbb{R}^2 \mapsto \mathbb{R}^2$ which preserves the lengths of the curves and maps the unit circle $|x| = 1$ inside the disk $|x| < 0$. The answer is no if you want U to be C^2 because there will be an obstruction at the level of the curvature (an observation made long time ago by Gauss). And the answer is yes if you just require $U \in C^1$ (which is necessary to define the length but allows infinite curvature , Nash (1974) Kuiper (1955)).


$\vec{U}$
R
r
γ
$U(\gamma)$

$$\begin{pmatrix} |\frac{\partial U}{\partial x_1}|^2, & \frac{\partial U}{\partial x_1}\cdot\frac{\partial U}{\partial x_2} \\ \frac{\partial U}{\partial x_1}\cdot\frac{\partial U}{\partial x_2} & |\frac{\partial U}{\partial x_2}|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Relax the equality for an inequality

$$\begin{pmatrix} |\frac{\partial U}{\partial x_1}|^2, & \frac{\partial U}{\partial x_1}\cdot\frac{\partial U}{\partial x_2} \\ \frac{\partial U}{\partial x_1}\cdot\frac{\partial U}{\partial x_2} & |\frac{\partial U}{\partial x_2}|^2 \end{pmatrix} \leq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Euler equation is equivalent to the linear system

$$\partial_t v + \nabla \cdot M + \nabla q = 0\,, \quad \nabla \cdot v = 0 \tag{1}$$

$$\text{with the constraint } M = v \otimes v - \frac{|v|^2}{d} I_d \tag{2}$$

$$\text{for } (v, M, q) \in L^\infty(\mathbb{R}_t \times \mathbb{R}^n_x; \mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R})$$

Next we relax the constraint and consider solution of (1) satisfying the estimate:

$$(v(x,t), M(x,t)) \in \mathbb{R}^n \times \mathcal{S}^n_0 \quad |v(x,t)|^2 \le 2e(x,t) \text{ and } v \otimes v - M \le \frac{2e(x,t)}{n} I_n$$

Show that this set is non empty and that the extremal points of its convex hull correspond to the relation (2) at this point.
As in the isometric imbedding (and as in the first proofs of Scheffer and Shnirelman) one constructs solutions with the accumulation oscillations, of small amplitude but high frequency, with a very careful analysis.

Viscosity solutions and Dissipative solutions

A term coined by P.L Lions and M. Crandall for solutions of equations with no viscosity but limit of solutions of equations with viscosity.

$$S(w)=\frac{1}{2}(\nabla w+(\nabla w)^t)\,,\quad \partial_t w+P(w\cdot\nabla w)=E(x,t)=E(w)$$

$$\partial_t u+\nabla\cdot(u\otimes u)+\nabla p=0\,,\nabla u=0\,,u\cdot\vec{n}=0 \text{ with } u \,\text{smooth}\,,$$

$$\partial_t w+w\cdot\nabla w+\nabla q=E(w)\,,\quad \nabla\cdot w=0\,.$$

$$\begin{aligned}&\frac{1}{2}\int_\Omega|u(x,t)-w(x,t)|^2\le\int_0^t\int|(E(x,s),u(x,s)-w(x,s))|dxds\\&+\int_0^t\int_\Omega|(u(x,s)-w(x,s)S(w)u(x,s)-w(x,s))|dxds\\&+\frac{1}{2}\int_\Omega|u(x,0)-w(x,0)|^2dx\,.\end{aligned}\tag{3}$$

A dissipative solution is as a divergence free tangent to the boundary vector field which for any test function w as introduced above satisfies the relation (3).

Hence the stability of dissipative solutions with respect to smooth solutions and, in particular, the fact that whenever exists a smooth solution $u(x,t)$ any dissipative solution which satisfies $w(.,0) = u(.,0)$ coincides with u for all time.

However, it is important to notice that to obtain this property one needs to include in the class of test functions w vector fields that may have non zero tangential component on the boundary.

Viscosity limit

$$\partial_t u_\nu + u_\nu \cdot \nabla u_\nu - \nu \Delta u_\nu + \nabla p_= 0\,, \quad \overline{u} = \text{weak} - \lim_{\nu \to 0} u\,,$$

$$\partial_t w - \nu \Delta w + w \cdot \nabla w + \nabla q = E(w)\,,$$

$$\frac{1}{2}\frac{d}{dt}|u_\nu(x,t) - w(x,t)|^2_{L^2(\Omega)} + \nu |\nabla u_\nu(t)|^2_{L^2(\Omega)}$$

$$\leq |(S(w) : (u_\nu - w) \otimes (u_\nu - w))| + |(E(w), u_\nu - w)|$$

$$+\nu(\nabla u_\nu, \nabla w)_{L^2(\Omega)} + \nu(\partial_{\vec{n}} u_\nu, u_\nu - w)_{L^2(\partial\Omega))}\,,$$

$$\frac{1}{2}\int_\Omega |\overline{u}(x,t) - w(x,t)|^2 \leq \int_0^t \int |(E(x,s), \overline{u}(x,s) - w(x,s))| dxds$$

$$+ \int_0^t \int_\Omega |(\overline{u}(x,s) - w(x,s) S(w) \overline{u}_\nu(x,s) - w(x,s))| dxds$$

$$+\frac{1}{2}\int_\Omega |u(x,0) - w(x,0)|^2 dx + \lim_{\nu \to 0} \nu \int_0^t \int_{\partial\Omega} (\partial_{\vec{n}} u_\nu, u_\nu - w) d\sigma dt$$

With no boundary convergence (modulo subsequence) to a dissipative solution is always true.

If there exists a smooth solution $u(x,t)$ on $[0,T]$ with the same initial data then $\overline{u}(x,t) = u(x,t)$.

$$\frac{1}{2}\int |\overline{u}(x,0)|^2 dx = \frac{1}{2}\int |\overline{u}(x,t)|^2 dx \leq \frac{1}{2}\int |\overline{u}_\nu(x,t)|^2 dx \leq \frac{1}{2}\int |\overline{u}(x,0)|^2 dx$$

- In the absence of boundary and with the existence of a smooth solution of the Euler equation there is no anomalous energy dissipation, no w-Reynolds stress tensor.
- In the absence of regular solution (loss of regularity for Euler solution or wild initial data) $\overline{u}$ is still a dissipative but may be not a weak solution (Reynolds stress tensor $\neq 0$ and may not be the unique solution. Non uniqueness of dissipative solutions for a large set of initial data.

Claim: It is in the presence of physical boundaries that the relation between turbulence and energy dissipation is the most evident .

A "general family " of boundary conditions containing the "classical":

$$u_\nu \cdot \vec{n} = 0 \quad \text{and}\,(\partial_{\vec{n}} u_\nu + (C(x)u_\nu)_\tau + \lambda(\nu)u_\nu = 0 \ \text{ on } \partial\Omega \qquad (4)$$
$$\text{with } \lambda(\nu, x) \geq 0 \quad \text{and} \quad C(x) \in C(\mathbb{R}^n, \mathbb{R}^n) \qquad (5)$$

$$u_\nu \cdot \vec{n} = 0 \Rightarrow ((\nabla^\perp u_\nu) \cdot \vec{n})_\tau = (C(x)u_\nu)_\tau$$

Hence with $u_\nu \cdot \vec{n} = 0$ are of the type (4):

$$\text{Dirichlet with } \lambda(\nu) = \infty\,,$$
$$\text{Dirichlet-Neumann with} \quad \lambda(\nu) = C(x) = 0\,,$$
$$\text{Fourier with } C(x)(u_\nu) = (\nabla^\perp u_\nu)) \Rightarrow \nu((S(u_\nu)\vec{n})_\tau + \lambda(\nu)u_\nu = 0\,,$$
$$\text{With vorticity} \quad \nu((\nabla(u_\nu) - \nabla^\perp(u_\nu))\vec{n})_\tau + \lambda(\nu)u_\nu = 0\,.$$

Energy estimates

$$\frac{1}{2}\int |u_\nu(x,T)|^2 dx + \int_0^T (\nu \int_\Omega |\nabla u_\nu|^2 dx + \int_{\partial\Omega} \lambda(\nu)|u_\nu(x,t)|^2 d\sigma) dt =$$
$$\frac{1}{2}\int |u_\nu(x,0)|^2 dx + o(\nu)$$
$$\frac{1}{2}\frac{d}{dt}|u_\nu(x,t) - w(x,t)|^2_{L^2(\Omega)} + \nu|\nabla u_\nu(t)|^2_{L^2(\Omega)}$$
$$\leq |(S(w) : (u_\nu - w) \otimes (u_\nu - w))| +$$
$$|(E(w), u_\nu - w)| + \nu(\partial_{\vec{n}} u_\nu, w)_{L^2(\partial\Omega))} + o(\nu)\,.$$

Theorem Convergence to a dissipative solution:

1 In any case, in particular Dirichlet $(\nu\frac{\partial u_\nu}{\partial \vec{n}})_\tau \to 0$ in $\mathcal{D}'(\partial\Omega\times]0,T[)$,

2 For Fourier-Navier $\lambda(\nu)u_\nu \to 0$: in $\mathcal{D}'(\partial\Omega\times]0,T[) \to 0$,

3 $\lambda(\nu)\to 0$ or $\lambda(\nu)$ bounded and $\int_{\partial\Omega\times]0,T[}\lambda(\nu)|u_\nu(x,t)|^2 d\sigma dt \to 0$,

4 In any case Kato $\lim_{\nu\to 0}\nu\int_0^T\int_{\Omega\cap\{d(x,\partial\Omega)<\nu\}}|\nabla u_\nu(x,t)|^2 dxdt \to 0$.

No turbulence in the presence of physical boundary

In the presence of a smooth solution u for Euler equation on $[0, T]$ with the same initial data the following facts are equivalents

- $w - \lim(u_\nu \otimes u_\nu) = (w. \lim u_\nu) \otimes (w - \lim u_\nu)$. No w-Reynold stress tensor.
- $u_\nu \rightharpoonup u$. Weak convergence to the solution of the Euler equations.
- $\forall 0 < t < T \ \frac{1}{2}\int_\Omega |w. \lim u_\nu(x,t)|^2 dx = \frac{1}{2}\int_\Omega |u_0(x)|^2 dx$. Energy conservation.
- $u_\nu \to u$. Strong convergence
- $\lim_{\nu\to 0} \nu \frac{\partial u_\nu}{\partial \vec{n}} = 0$ in $\mathcal{D}'(\Omega)$. No anomalous vorticity production at the boundary.
- $\lim_{\nu\to 0} \int_0^T (\int_\Omega \nu |\nabla u_\nu(x,t)|^2 dx + \lambda(\nu) \int_{\partial\Omega} |u_\nu|^2 d\sigma) dt = 0$. No anomalous energy dissipation.
- $\lim_{\nu\to 0} \int_0^T \int_{d(x,\partial\Omega)<\nu} |\nabla u_\nu(x,t)|^2 dx = 0$. No anomalous "order ν" boundary layer energy dissipation.

Remarks

• The existence of a Prandlt boundary layer implies Kato hypothesis. (converse may not be true).

• If one of the above equivalent fact is not satisfied one would expect generation of turbulence.

The limit is not a solution of the Euler equations, there is no energy conservation, there is anomalous energy dissipation, the weak Reynolds stress tensor is not 0 .

Turbulence spectra and Wigner Measure

Eventually the weak Reynolds tensor is defined in term of Wigner Measure

$$\lim_{\nu\to 0}((u_\nu \otimes u_\nu) - (\overline{u}\otimes\overline{u})) = \lim_{\nu\to 0}(u_\nu - \overline{u})\otimes(u_\nu - \overline{u}) = \int_{\mathbb{R}^n} W(x,t,\xi)d\xi$$

$$W(x,t,\xi) = \lim_{\nu\to 0} W_\nu(x,t,\xi)$$

$$W_\nu(x,t,\xi) = (\frac{1}{2\pi})^d \int_{\mathbb{R}^n} u_\nu(x+\frac{\sqrt{\nu}y}{2},t)\otimes u_\nu(x-\frac{\sqrt{\nu}y}{2},t)e^{iy\cdot\xi}dy$$

$$-(\overline{u}(x,t)\otimes\overline{u}(x,t))$$

The turbulence spectra!!!

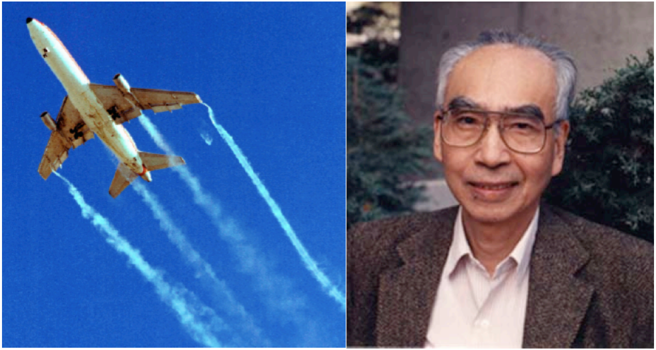


Figure: Kato: Prandlt..Boundary layer, Kelvin Helmholtz, Von Karman vortex street.

Proof of Kato argument

For any $w \in T(\partial\Omega\times]0,T[)$ introduce a sequence $w_\nu(s,\tau,t)$ (in geodesic coordinates near $\partial\Omega$) with

$$\mathrm{support}(w_\nu) \subset \Omega_\nu\times]0,T[\,, \nabla\cdot w_\nu = 0, \text{ and on } \partial\Omega\times]0,T[\quad w_\nu = w\,,$$
$$|\nabla_{\tau,t} w_\nu|_{L^\infty} \le C\,, \quad |\partial_s w_\nu|_{L^\infty} \le \frac{C}{\nu}\,.$$

From

$$(0,w_\nu) = ((\partial_t u_\nu + \nabla(u_\nu\otimes u_\nu) - \Delta u_\nu + \nabla p_\nu)w_\nu) =$$
$$-(u_\nu,\partial_t w_\nu) + ((u_\nu\otimes u_\nu):\nabla w_\nu) + \nu(\nabla u_\nu,\nabla w_\nu) - (\nu\partial_{\vec n} u_\nu w)_{L^2(\partial\Omega\times]0,T[}$$
$$\Rightarrow |(\nu\partial_{\vec n} u_\nu w)_{L^2(\partial\Omega\times]0,T[)}| = |((u_\nu\otimes u_\nu):\nabla w_\nu)| + o(\nu)$$

Poincaré estimate and a priori estimate

$$\Rightarrow |((u_\nu\otimes u_\nu):\nabla w_\nu)| \le C\int_0^T\int_{\Omega_\nu}\nu|\nabla u_\nu|^2 dxdt \to 0\,.$$

More comments about 2 Euler equations

The $2d$ Euler equation describes planar flows and as such the vorticity is conserved along the trajectories of the particles (Lagrangian coordinates). Therefore (that can be observed by direct computation one has)

$$\partial_t\omega + u\cdot\nabla\omega = 0\,.$$

This may bring more informations but no simplifications for many of the problems considered above.

More comments about 2 Euler equations

- It was proven by Wolibner, Kato, Youdovitch that with initial data in $C^{1,\alpha}$ there is a unique solution defined for all time which is as smooth as the initial data.

However this regularity may deteriorate with time (for instance if the initial data $u_0(x)$ has 2 bounded derivatives same will be for the solution at any time $t > 0$ but there is no proof that these derivatives have an exponential growth (as it would be the case for system solutions of classical dynamical systems) the only thing that can be proven is a control by a double exponential $exp(expKt)$.

The reason is that an L^∞ control of the vorticity does not implies an L^∞ control of all the gradient ∇u.

More comments about 2 Euler equation

- One can prove easily the existence of weak solutions with initial data having vorticity no more singular than a positive measure (or a measure with a "reasonable " number of change of sign).

Such solutions would be a vorticity solutions has defined above.

- On the other hand all the results concerning wild solutions described above are as valid in $2d$ as in $3d$. Therefore it is fully open to understand what type of regularity on the vorticity would ensure uniqueness and some type of stability.

$\omega_0 \in L^\infty$ implies existence of a unique weak solution. This is not true for $u_0 \in L^2$ with no extra hypothesis on the vorticity. Is there any relation between the wild weak solution and the weak solution constructed say with $\omega_0 \in L^1$

- All the issues relating the relation beetwen boundary effects and turbulence as described above in $3d$ are similar in $2d$.

More comments about 2 Euler equations

- The discussion about weak solutions has more importance than a game for mathematicians!! In particular when one consider in $2d$ planar fluids with vorticity concentrated on a curve.

This configuration carries the name of Kelvin-Helmholtz problem and is currently used to describe experiments or numerical simulation

What is known is:

1 That an initial data with a very smooth (analytical) vorticity concentrated on a very smooth (analytical) curve leads to an unstable local in time solution described by a integral equation called the Birkoff-Rott equation.

2 That an initial data with vorticity concentrated on a curve will lead to a smooth solution (defined for all time) solution of the Navier Stokes equation with any positive viscosity $\nu > 0$.

More comments about 2 Euler equation

3 That when ν goes to 0 the solution converges (with minor hypothesis on the change of sign of the vorticity on the curve) to a dissipative-viscosity solution of the Euler equations defined for all time.
However due to the lack of uniqueness no one knows if this weak limit coincides with the solution of the Birkoff Rott equation when it is defined (for a finite time).
An explicit example the Prandlt-Munk vortex shows that the weak limit may not be, in some singular cases, described by the Birkoff-Rott equation

Boltzmann →Euler limit with boundary effect

To consolidate the fact that Kato approach may be the correct point of view and that the boundary condition

$$\nu(\partial_{\vec{n}} u_\nu + (C(x) u_\nu)_\tau + \lambda(\nu) u_\nu = 0$$

(which contains Dirichlet and Neumann) is the good one one can argue that the introduction of a microscopic derivation based on the Boltzmann equation leads to the same results.

$F_\epsilon(x,v,t)\geq 0$: Density of particles which at the point $x\in\Omega$ and the time t do have the velocity $v\in\mathbb{R}^n_v$) of the (rescaled in time) Boltzmann equation:

$$\epsilon\partial_t F_\epsilon + v\cdot\nabla_x F_\epsilon = \frac{1}{\epsilon^{1+q}}\mathcal{B}(F_\epsilon,F_\epsilon)\ \text{Quadratic operator in } \mathbb{R}^n_v$$

with Maxwell Boundary Condition for $v\cdot\vec{n}<0$ in term of $v\cdot\vec{n}>0$:

$$F_\epsilon^-(x,v)=(1-\alpha(\epsilon))F_\epsilon^+(x,v^*)+\alpha(\epsilon)M(v)\sqrt{2\pi}\int_{v\cdot\vec{n}<0}|v\cdot\vec{n}|F_\epsilon^+(x,v)dv\,,$$

$$0\leq\alpha(\epsilon)\leq 1\,,\, v^*=v-2(v\cdot\vec{n})\vec{n}=\mathcal{R}(v)\,,$$

$$M(v)=\frac{1}{(2\pi)^{\frac{n}{2}}}e^{-\frac{|v|^2}{2}}\,,\quad \Lambda(\phi)=\sqrt{2\pi}\int_{\mathbb{R}^n_v}(v\cdot\vec{n})_+\phi(v)M(v)dv,$$

$$\Lambda(1)=1(\text{proba!})\quad F_\epsilon^-(x,v)=(1-\alpha(\epsilon))F_\epsilon^+(x,\mathcal{R}(v))+\alpha(\epsilon)\Lambda(\frac{F_\epsilon}{M})\,,$$

$$F_\epsilon(x,v,0)=M(v)(1+\epsilon g(v))\quad \lim_{\epsilon\to 0}u_\epsilon=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\mathbb{R}^n_v}\vec{v}F(\epsilon(x,v,t)dv\,.$$

• For $q = 0$, $u_\epsilon = \frac{1}{\epsilon}\int_{\mathbb{R}^n_v} vF_\epsilon dv$ converges to a Leray solution of Navier-Stokes with the boundary condition:

$$u\cdot\vec{n} = 0 \quad \text{and} \quad \nu((\nabla u + \nabla^t u)\cdot n)_\tau + \lambda(\nu)u = 0$$
$$\lambda(\nu) = \frac{1}{\sqrt{2\pi}}\lim_{\epsilon\to 0}\frac{\alpha(\epsilon)}{\epsilon} \quad \text{Dirichlet} \Leftrightarrow \lim_{\epsilon\to 0}\frac{\alpha(\epsilon)}{\epsilon} = \infty\,.$$

Entropy Dissipation versus Energy Balance

$$H(F|G) = \int_{\Omega\times\mathbb{R}^n_v} (F\log(\frac{F}{G}) - F + G)dxdv \text{ Relative entropy },$$

$$\frac{1}{\epsilon^2}\frac{d}{dt}H(F_\epsilon(t)|M) + \frac{1}{\epsilon^{q+4}}\int_\Omega\int_{\mathbb{R}^3_v} DE(F_\epsilon)dvdv_1d\sigma + \frac{1}{\epsilon^3}\int_{\partial\Omega}\mathrm{DG} = 0$$

$$\mathrm{DE}(F)(v,v_1,\sigma) = \frac{1}{4}(F'F'_1 - FF_1)\log(F'F'_1 - FF_1)b(|v-v_1|,\sigma) \text{ En. diss}$$

$$\mathrm{DG}(F) = \int_{\mathbb{R}^3_v} v\cdot\vec{n}H(F_\epsilon|M)d\sigma dv \text{ The Darrozes-Guiraud local entropy }.$$

$$h(z) = (1+z)\log(1+z) - z)$$

$$\sqrt{2\pi}\mathrm{DG} = \int_{\mathbb{R}^3_v} v\cdot\vec{n} H(F_\epsilon|M)d\sigma dv =$$

$$\sqrt{2\pi}\int_{\mathbb{R}^3_v} v\cdot\vec{n} H(M(1+\epsilon g_\epsilon)|M)dv = \sqrt{2\pi}\int_{\mathbb{R}^3_v} v\cdot\vec{n} M(v)h(1+\epsilon g_\epsilon)dv$$

$$= \sqrt{2\pi}\int_{\mathbb{R}^3_v}(v\cdot\vec{n})_+ M(v)h(\epsilon g_\epsilon(v))dv - \sqrt{2\pi}\int_{\mathbb{R}^3_v}(v\cdot\vec{n})_+ M(v)h(\epsilon g_\epsilon(\mathcal{R}v))$$

$$= \Lambda(h(\epsilon g_\epsilon)) - \Lambda(h[(1-\alpha(\epsilon))\epsilon g_\epsilon + \alpha(\epsilon)\Lambda(\epsilon g_\epsilon)])$$

$$\geq \alpha(\epsilon)\Big[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v)))\Big] \geq 0$$

Hence the final entropy estimate:

$$\frac{1}{\epsilon^2}\frac{d}{dt}H(F_\epsilon(t)|M)+\frac{1}{\epsilon^{q+4}}\int_\Omega\int_{\mathbb{R}^3_v}DE(F_\epsilon)dvdv_1d\sigma$$
$$+\frac{1}{\epsilon^2}\frac{\alpha(\epsilon)}{\epsilon}\frac{1}{\sqrt{2\pi}}\int_{\partial\Omega}[\Lambda(h(\epsilon g_\epsilon(v)))-h(\Lambda(\epsilon g_\epsilon(v)))]d\sigma\leq 0\,.$$

Compare formally to energy with $g_\epsilon = \epsilon^{-1}(F_\epsilon - M)/M \to u \cdot v$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\nu(x,t)|^2 dx + \nu \int_\Omega |\nabla u_\nu|^2 dx + \int_{\partial\Omega} \lambda(\nu)|u_\nu(x,t)|^2 d\sigma \to 0$$

$$\frac{1}{\epsilon^2}\frac{d}{dt}H(F_\epsilon(t)|M) \to \frac{1}{2}\frac{d}{dt}\int_\Omega |u(x,t)|^2 dx$$

$$\frac{1}{\epsilon^{q+4}}\int_\Omega\int_{\mathbb{R}^3_v} DE(F_\epsilon) dv dv_1 d\sigma \simeq \epsilon^q \nu \int_\Omega |\nabla u + \nabla^\perp u|^2 dx$$

$$\frac{1}{\epsilon^2}\int_{\partial\Omega} [\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v)))] d\sigma \simeq \int_{\partial\Omega} |u_\epsilon(x,t)|^2 d\sigma$$

$$\frac{\alpha(\epsilon)}{\epsilon}\frac{1}{\sqrt{2\pi}} \simeq \lambda(\epsilon^q \nu)$$

Entropic convergence to a regular Euler solution $\Rightarrow$

$$\frac{1}{\epsilon^{q+4}}\int_\Omega\int_{\mathbb{R}^3_v} DE(F_\epsilon)dvdv_1d\sigma$$
$$+\frac{1}{\epsilon^2}\frac{\alpha(\epsilon)}{\epsilon}\frac{1}{\sqrt{2\pi}}\int_{\partial\Omega}[\Lambda(h(\epsilon g_\epsilon(v)))-h(\Lambda(\epsilon g_\epsilon(v))))]d\sigma\to 0$$

Theorem Sufficient condition for the convergence to Euler:

$$\lim_{\epsilon\to 0}\frac{\alpha(\epsilon)}{\epsilon}=0 \text{ or}$$
$$\frac{\alpha(\epsilon)}{\epsilon}\le C<\infty \text{ and } \frac{1}{\epsilon^2}\int_{\partial\Omega\times]0,T[}[\Lambda(h(\epsilon g_\epsilon(v)))-h(\Lambda(\epsilon g_\epsilon(v))))]d\sigma dt\to 0$$

Conjecture (Kato!)

$$\frac{1}{\epsilon^{q+4}}\int_0^T\int_{\Omega\cap\{d(x,\partial\Omega)\le\epsilon^q)\}}\int_{\mathbb{R}^3_v} DE(F_\epsilon)dvdv_1d\sigma dt\to 0\,.$$

Conclusion

• The main observation of this talk is the fact that it is in the analysis of the limit (for $\Re e \to \infty$) of problems with boundary effect that there is the most evident similarity between a deterministic and a statistical theory of turbulence.

• What can be shown is that in this situation the non existence of turbulence in the fluid is characterized by a serie of equivalent properties.

• This means that turbulence would be characterized by violation of any of these properties (conservation of energy, zero Reynolds stress tensor, no production of vorticity in a very small boundary layer etc...)

Conclusion

- In the course of the analysis it is important to keep in mind recent mathematical progress on the Euler equations to contribute to the issue of the weak limit.
- Even if in the present formulation Euler equation have no physical relevance. With the d'Alembert paradox they would imply that plane cannot fly and which the recent works on wild solutions they would solve the energy crisis.
- The introduction of the Boltzmann equation at the end of the talk is done to consolidate the point of view given previously on the effects of the boundary.