SECTION

TURBULENCE EN MILIEUX COMPRESSIBLES ET ÉLECTROCONDUCTEURS

TURBULENCE IN COMPRESSIBLE AND ELECTRICALLY CONDUCTIVE MEDIA

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INTRODUCTION

La recherche en turbulence semble présenter deux possibilités d'approche plutôt contradictoires, ou, usant de termes plus modérés, complémentaires. Dans l'une, un problème de plus en plus simplifié est posé, de sorte que finalement, une seule complication essentielle est retenue, et tous les efforts sont alors concentrés pour la compréhension de ce problème très simplifié.

La turbulence homogène et isotrope d'un fluide incompressible à un nombre de Reynolds élevé est ainsi un tel problème « par excellence » (et il est encore loin d'une compréhension totale).

L'autre moyen d'approche est l'exploration, même « en gros » seulement, ou d'une façon grossière, de nombreuses sortes de complications nouvelles telles que la présence d'un gradient moyen, d'une réaction chimique, d'une force de masse, ou, comme c'est le cas ici, l'introduction de la compressibilité ou de la conductivité électrique. Cela étant, ce second mode d'approche naît à la fois de la curiosité pour des phénomènes nouveaux, et de l'impatience due à la lenteur du progrès réalisé vers une pleine compréhension, même des problèmes les plus simples.

Bien sûr, l'espèce de « compréhension » qui résulte de tels efforts moins pénétrants n'est évidemment que partielle.

Les rôles relatifs de ces deux sortes d'investigation sont quelque peu analogues à ceux familiers, communément usités dans les sciences de la vie, où l'on peut se saisir des problèmes à « différents niveaux de leur organisation », c'est-à-dire que l'on n'attend pas nécessairement jusqu'à ce que tout soit entièrement résolu à un niveau plus simple, plus élémentaire. (Par exemple : on n'attend pas pour résoudre les problèmes biochimiques que tout le mécanisme de la chimie ait été déduit de la physique de l'atome).

Dans un ordre d'idées tout à fait comparable, il semble qu'il vaille la peine de poser des questions concernant les grandes lignes d'un phénomène tel que la turbulence en écoulement compressible, ou la turbulence d'un fluide conducteur de l'électricité en présence d'un champ magnétique intense, longtemps avant que la théorie complète et satisfaisante du mécanisme de transfert d'énergie en turbulence incompressible et isotrope ne soit totalement achevée.

En fait, il est possible de rapporter quelques progrès réels faits dans ces nouvelles tentatives, et le but de la session spéciale d'aujourd'hui est de discuter le genre de progrès que ces problèmes ont réalisé jusqu'ici.

Lorsqu'on a pu faire entrer en ligne de compte des propriétés supplémentaires nouvelles, il existe deux questions typiques sur le comportement général d'un phénomène.

La première peut être posée ainsi :

Dans quelle mesure les aspects usuels déjà établis seront-ils modifiés, et de quelle façon les solutions seront-elles transformées ?

Un exemple assez classique est l'effet de la compressibilité dans la théorie de l'aile. Dans ce cas, on répond à la question en établissant une « correction de compressibilité ».

Dans de nombreux cas, la seconde question est de loin plus intéressante :

Y a-t-il apparition de phénomènes supplémentaires entièrement nouveaux qui n'existaient pas du tout avant l'introduction de nouvelles propriétés ?

L'apparition des ondes de choc en fluide compressible constitue un exemple classique, et l'apparition d'une nouvelle sorte d'ondes se propageant, les ondes d'Alfvén, en magnéto-dynamique des fluides en est un autre.

En ce qui concerne les écoulements turbulents, les deux types de questions sont légitimes, et significatives. Bien sûr, l'effet de la compressibilité a été étudié plus longtemps, et on dispose de documents expérimentaux plus nombreux.

La recherche sur la turbulence en magnéto-dynamique des fluides en est encore à un stade de développement plus spéculatif, et il n'existe pratiquement pas de mesures publiées pour confirmer les idées présentées ici.

Turbulence research appears to have two rather contradictory or, using a milder term, two complementary modes of approach. In one, a more and more simplified problem is posed so that finally only one essential complication is retained and then all efforts are concentrated on understanding that oversimplified problem. Homogeneous isotropic turbulence of an incompressible fluid at high Reynolds number is such a problem " par excellence " (and it is still far from complete understanding). The other type of approach is the exploration, even if only " in the large ", or in a crude way, of many new kinds of complications such as the presence of a mean gradient or of a chemical reaction or of a body force, or, as in the present case, the inclusion of compressibility or of electrical conductivity. Admittedly, this second approach is born both out of curiosity for new phenomenon and out of impatience with the slow progress made toward full understanding even the simplest problem. Of course, the kind of " understanding " derived from such less penetrating efforts is clearly only a partial one. The relative roles of the different approaches are somewhat analogous with the familiar ones common in the life sciences where one may tackle the problem at " different levels of organization ", meaning that one does not necessarily wait until everything is completely solved on the simpler, more elementary level. (E.g., one does not wait to solve biochemical problems until all chemical behavior has been derived from atomic physics.)

In quite a similar vein, it appears worthwhile to ask questions about the gross features of such phenomena as compressible flow turbulence or the turbulence of an electrically conductive fluid in the presence of strong magnetic fields, long before the complete and satisfactory theory is completed for the energy transfer mechanism in incompressible isotropic turbulence.

As a matter of fact, it is possible to report some actual progress made in these new attempts and the purpose of today's special session is to discuss the kind of progress these problems have so far attained.

When new additional properties have been permitted to come into play, there are two typical questions about the general behavior of a phenomenon. The first can be posed this way : How much the old, already well-established features will be modified, in what way will the solutions be " warped " ? A rather classic example is the compressibility effect in airfoil theory. In this case the question in answered by developing a " compressibility correction ".

In many ways the second type of question is far more interesting : Will entirely new, additional phenomena appear that were not present at all before the new properties were included ? The appearance of shock waves in a compressible flow is the classic example, and the appearance of a new kind of propagating wave, the Alfvén wave, in magneto-fluid-dynamics is another one.

As far as turbulent flows are concerned, both types of questions are legitimate and meaningful. Of course, the effect of compressibility has been studied longer and there is more experimental evidence available. Research in magneto-fluid-dynamic turbulence is still in a more speculative stage of development and there are virtually no measurements published to confirm the ideas presented here.

Effect of compressibility

Incompressible turbulence is a random solenoidal velocity field with a concommittant pressure fluctuation field. The pressure field is, however, completely determined by the velocity fluctuation field and the controlling variable of the turbulence is the random vorticity field. Random temperature fluctuations in low speed flow have been treated as a transport problem by regarding temperature as a " passive contaminant " without actually considering any dynamic effect on the turbulence itself.

The effect of compressibility has been studied by inspecting the " weak field " limit of the Navier-Stokes equations (Kovasznay, 1953) (Chu and Kovásznay, 1958). The governing equations for a compressible viscous heat-conductive gas are nonlinear partial differential equations. Even if the transport properties (viscosity and heat conductivity) are assumed to be constant there are still six dependent variables (three components of the velocity, the pressure, the density and the temperature) of the gas governed by the six equations (conservation of mass, conservation of the three components of momentum, conservation of the energy and the equation of state for the gas). The weak field limit (linearization around the " rest solution " $\mathbf{u} \equiv 0$, $\rho =$ const, $p =$ const, $T =$ const) makes the decoupling of three independent " modes " possible. These are (with some simplifications.

Vorticity (or solenoidal) mode :

$$\frac{\partial \boldsymbol{\omega}^{(1)}}{\partial t} - \nu \nabla^2 \boldsymbol{\omega}^{(1)} = 0$$

Pressure or acoustic mode :

$$\nabla^2 P^{(1)} - \frac{1}{a^2} \frac{\partial^2 P^{(1)}}{\partial t^2} = 0 \qquad (2)$$

Entropy mode :

$$\frac{\partial S^{(1)}}{\partial t} - \frac{4\nu}{3} \nabla^2 S^{(1)} = 0 \qquad (3)$$

(The superscript $^{(1)}$ emphasizes that it is a first order approximation.)

Equation (1) represents the solenoidal velocity field that can be identified with the incompressible turbulence. Equation (2) governs acoustic wave propagation and also includes all irrotational velocity fields. The third mode (3) can be regarded as a random temperature field. The three modes do not interact among themselves within the framework of the linear theory, except through boundary conditions on solid walls, because they are imposed on the velocity and temperature, not on the separate modes and in general more than one mode contributes to both temperature (P and S) and to velocity (ω and P).

Within the framework of the linearized theory, there is no energy transfer from one wavelength to another, therefore the usefulness of such an approach is automatically limited to such things as the interpretation of measurements at a point or it may be conceptually useful as the first step in a systematic expansion scheme. The complete second order theory has been developed by CHU and KOVASZNAY (1958). The three linearized modes are " driven " by the nonlinear interactions and a complete expansion scheme was developed. In essence the governing equations have the left-hand side as in (1), (2) and (3), but the right-hand side contains a " driving " term, a bilinear expression of the three modes, always formed with the lower (first) order solutions.

If we put $\boldsymbol{\omega} = \boldsymbol{\omega}^{(1)} + \boldsymbol{\omega}^{(2)}$; $P = P^{(1)} + P^{(2)}$; $S = S^{(1)} + S^{(2)}$, the vorticity mode equation becomes.

$$\frac{\partial \boldsymbol{\omega}^{(2)}}{\partial t} - \nu \nabla^2 \boldsymbol{\omega}^{(2)} = [\omega^{(1)}; \omega^{(1)}]_\omega + [P^{(1)}; P^{(1)}]_\omega + [S^{(1)}; S^{(1)}]_\omega + + [\omega^{(1)}; P^{(1)}]_\omega + [P^{(1)}; S^{(1)}]_\omega + [S^{(1)}; \omega^{(1)}]_\omega \tag{4}$$

The pressure mode equation now becomes

$$\nabla^2 P^{(2)} - \frac{1}{a^2}\frac{\partial^2 P^{(2)}}{\partial t^2} = [\omega^{(1)}; \omega^{(1)}]_P + [P^{(1)}; P^{(1)}]_P + [S^{(1)}; S^{(1)}]_P + + [\omega^{(1)}; P^{(1)}]_P + [P^{(1)}; S^{(1)}]_P + [S^{(1)}; \omega^{(1)}]_P \tag{5}$$

and finally the entropy mode equation

$$\frac{\partial S^{(2)}}{\partial t} - \frac{4\nu}{3} \nabla^2 S^{(2)} = [\omega^{(1)}; \omega^{(1)}]_S + [P^{(1)}; P^{(1)}]_S + [S^{(1)}; S^{(1)}]_S + + [\omega^{(1)}; P^{(1)}]_S + [P^{(1)}; S^{(1)}]_S + [S^{(1)}; \omega^{(1)}]_S \tag{6}$$

where the square brackets represent bilinear expressions in terms of the known ω, P and S each such expression acting as " source " term generating the particular mode that is indicated by the subscript ω, P or S. The expansion scheme can be followed up to any order and the left-hand side of the equation is always in the same form, only the explicit form of the nonlinear expressions becomes more and more involved. (E. g., in the third order equations both $\omega^{(1)}$ and $\omega^{(2)}$ appear). The second order theory has been developed quite completely and the second order (bilinear) interactions have been all identified. (CHU and KOVÁSZNAY, 1958). Just to mention the most interesting ones : the generation of the acoustic mode by the double interaction of the solenoidal velocity field (vorticity mode) $[\omega^{(1)}; \omega^{(2)}]_P$ (LIGHTHILL, 1952) and the generation of vorticity by the bilinear interaction (actually a vector cross-product of the density (entropy) gradient with the pressure gradient) $[P^{(1)}; S^{(1)}]_\omega$ (BJERKNES term).

In one respect even the second order theory falls short for the compressible turbulent boundary layer or in other shear flows. Second order interactions (double products in the dependent variables) may transfer energy from one mode to another or can scatter waves of one type on a disturbance of another type, but the transfer of energy from the mean motion into the turbulent motion, requires third order terms. This can be seen clearly from the structure of the turbulent energy equation (REYNOLDS equation)

$$\frac{1}{2}\frac{D}{Dt}[\overline{u_i' u_i'}] = \overline{u_i' u_\kappa'}\frac{\partial \overline{U}_i}{\partial x_\kappa} + \text{viscous and diffusion terms.} \tag{7}$$

Similar equations can be developed for the generation of mean square entropy fluctuations by " scrambling " a mean entropy gradient. The essential point is that in the mode structure also, the presence of a mean gradient must be regarded as a separate (even though steady) mode and the generating term on the right-hand side of eq. (7) is of third order in the velocity disturbances. Since the turbulent fluctuation levels are usually one order of magnitude smaller than the corresponding mean flow variables, higher order interactions (double or triple products) containing one mean flow gradient may overshadow the corresponding terms formed as the product of fluctuating quantities alone. A good example is the sound generation by turbulent shear flows. It was first pointed out by LIGHTHILL (1952) that the mean velocity gradients contribute much more to the generation of sound than the double products formed by the fluctuating velocities alone.

In dealing with compressible turbulence a great deal of work was done on sound generation. One must emphasize, however, that in the cases treated so far the sound must be regarded more as a by-product of the turbulent flow, than an integral part of the dynamics of turbulence. In other words, the sound energy radiated away from the turbulent region represents only a small fraction of the kinetic energy flux of turbulence and it is even small compared to the total viscous dissipation. One expects that the ratio of sound radiation loss to the viscous dissipation will increase as the MACH number increases and crude guesses suggest that the two energy losses may become equal around $M \approx 5$. In low supersonic MACH numbers ($M < 2$) the sound generation is quite small although the sound level on the rather arbitrary scale of the human ear (decibels) may be still quite offensive.

Magneto-fluid dynamic turbulence

Turbulence in an electrically conductive fluid became the subject of interest spurred by recent advances in astrophysics as well as in plasma experiments aimed at attaining magnetic confinement at a high temperature.

Our point of view is to regard the fluid as a continuum and no consideration on the particle level will enter directly. All microscopic phenomena are taken into account only globally as " bulk " properties, such as viscosity, electrical (scalar) conductivity, etc. This approach is probably valid for plasmas that are dense enough so that both the DEBYE shielding length and the ion cyclotron radius are much smaller than the smallest length scale of the turbulent motion (KOLMOGOROFF scale) and the plasma frequency is also much higher than the typical frequencies of the turbulence

$$l_K \gg l_D \qquad l_D \approx 6{,}9\sqrt{\frac{T_e}{n_e}}$$

$$\frac{u'}{\lambda} \ll f_P \qquad f_P \approx 9000\sqrt{n_e}$$

where

u' r. m. s. turbulent velocity fluctuations in cm/sec
l_K KOLMOGOROFF length in cm
l_D DEBYE shielding length in cm
λ microscale in cm
f_P plasma frequency in cps
n_e electron density in cm^{-3}
T_e electron temperature in °K

In liquid metals the molecular scale is very small : on the other hand, the very large magnetic viscosity is a great obstacle to experimentation. The magnetic kinematic viscosity is defined as

$$\nu_M = \frac{1}{\mu\sigma}$$

where μ is the magnetic permeability and σ is the electrical conductivity.

In general, it is difficult to achieve in the laboratory a sufficiently large magnetic Reynolds number (even to reach unity) to study turbulent phenomena in a regime where the electromagnetic effects on turbulence would be significant. For the known liquid metals (mercury, sodium and potassium) the ratio of viscosities is high $\left(\frac{\nu_M}{\nu} > 10^3\right)$.

In high temperature plasmas this ratio can be much more favorable but the unsteady nature of all known plasma experiments prevented so far drawing any quantitative conclusion about turbulence. Nevertheless, plasma experiments have been performed that at least suggest the presence of turbulence. When a plasma column is compressed rapidly by a magnetic field at some stage it ceases to behave according to the theoretical (laminar type) prediction and exhibits strong, unsteady disturbances that appear as more or less random fluctuations. These have been termed as instability, flutter, or turbulence according to the temperament (and vocabulary) of the different experimenters.

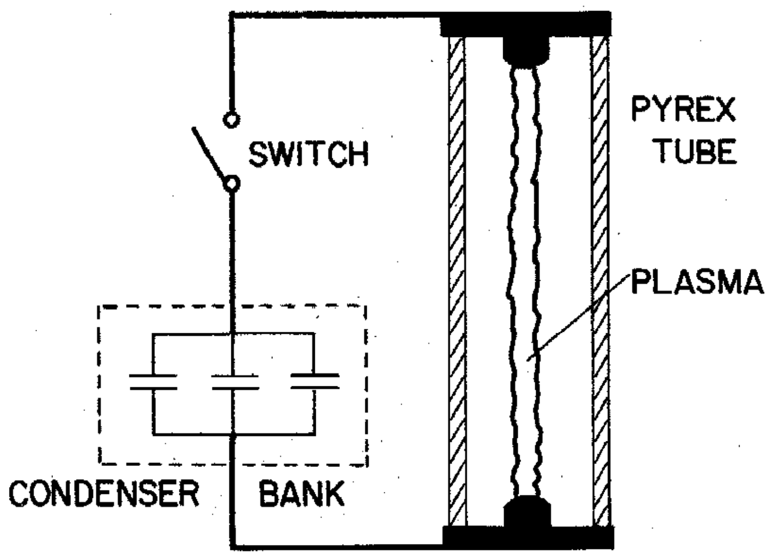


FIGURE 1
" Pinch " tube experiment.

One such experiment has been devised explicitly for turbulence studies and is shown in Fig. 1. A condenser bank (typically 500-1500 Joule stored energy) is discharged through a switch into a pyrex tube (typically 10 cm diameter and 50 cm length). In order to minimize the inductance of the configuration, the return current is conducted back coaxially (in the photograph shown in Fig. 3*a*, *b*, *c*, eight metal straps were used instead of a cylinder in order to give unobstructed view of the random pattern at least between the straps). The working fluid was hydrogen (or deuterium) gas at a " cold pressure " of 50-500 μHg before the discharge. The electric discharge (the current was of the order

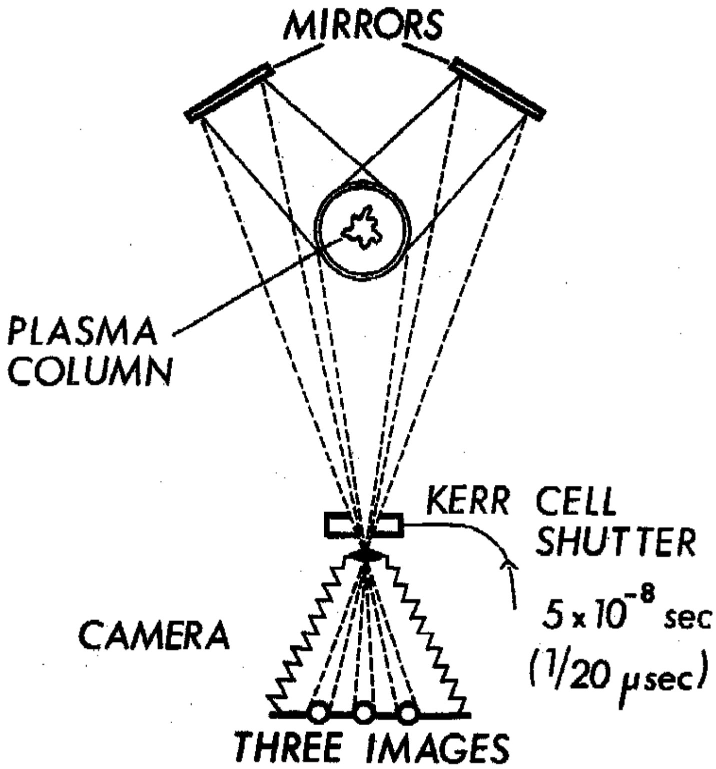


FIGURE 2
Kerr-cell shutter photography of plasma column.

of 100,000 - 150, 000 Amp) produced a plasma column and the induced azimuthal magnetic field (B_θ) caused a violent pinch effect and a cylindrical shock wave rushed radially inward. After attaining a minimum diameter (about 1/10 of the tube diameter) the column rebounds again, but then it appears randomly disturbed : it may be termed as turbulent.

Kerr-cell photographs with an exposure time of 5×10^{-8} sec were made with different time delays showing the phenomenon in different phases of development. In order to obtain a three-dimensional view of the discharge, two more images were taken by using two first surface mirrors (the disposition is given in Fig. 2) and the results are shown in Figs. *3a, b, c.* The three photographs give the general development of the plasma column at three different stages, the instant being 3.4, 4.0, and 4.4 microsecond after the discharge was initiated. A larger scale random pattern is clearly visible. It is also evident that the pattern is not periodic as predicted by the conventional instability theories. It is quite possible, however, that turbulence was created at the strongest pinch condition when the plasma had maximum density and the intensity of the magnetic field was also the highest. In the later development when the plasma expanded and cooled off again the only visible effect was that the random patterns of total luminosity seemed to have increased in scale *. Magnetic probes have been used successfully to record random fluctuations in the magnetic field when the plasma column appeared " turbulent ".

* The experiments were performed in collaboration with Dr. E. B. TURNER at Aerospace Corp., Los Angeles, California and author is grateful for permission to include these preliminary experimental results.

The theoretical problem of turbulence in a conductive fluid can be regarded as follows. There are two "energy-bearing" vector fields, the velocity field and the magnetic field. Their governing equations are somewhat similar but not identical and there is coupling between the two systems. The momentum equation governs **u** and it also contains the LORENTZ force term

$$\mathbf{J} \times \mathbf{B} = \mu (\nabla \times \mathbf{H}) \times \mathbf{H} \qquad (\text{for } \mu = \text{const}) \tag{8}$$

that is clearly nonlinear (quadratic) in the magnetic field vector **H**. The equation governing the magnetic field on the other hand, contains the velocity **u** but that occurs only linearly

$$\frac{\partial \mathbf{H}}{\partial t} + \nabla \times (\mathbf{H} \times \mathbf{u}) = \nu_M \nabla^2 \mathbf{H} \tag{9}$$

If this equation is solved formally

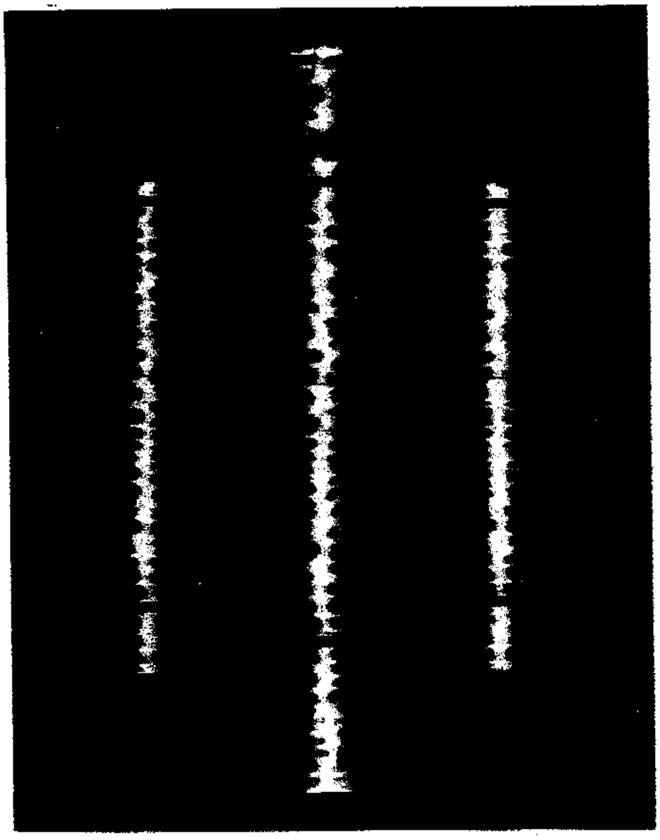
$$\mathbf{H} = \mathbf{L}(\mathbf{u}) + \mathbf{H}_0$$

where $\mathbf{L}(\mathbf{u})$ is a linear function of the velocity field and $\mathbf{H}_0$ is a solution of the corresponding homogeneous equation, then by substituting in eq. (8) we see that the LORENTZ force becomes a quadratic function of the velocity field

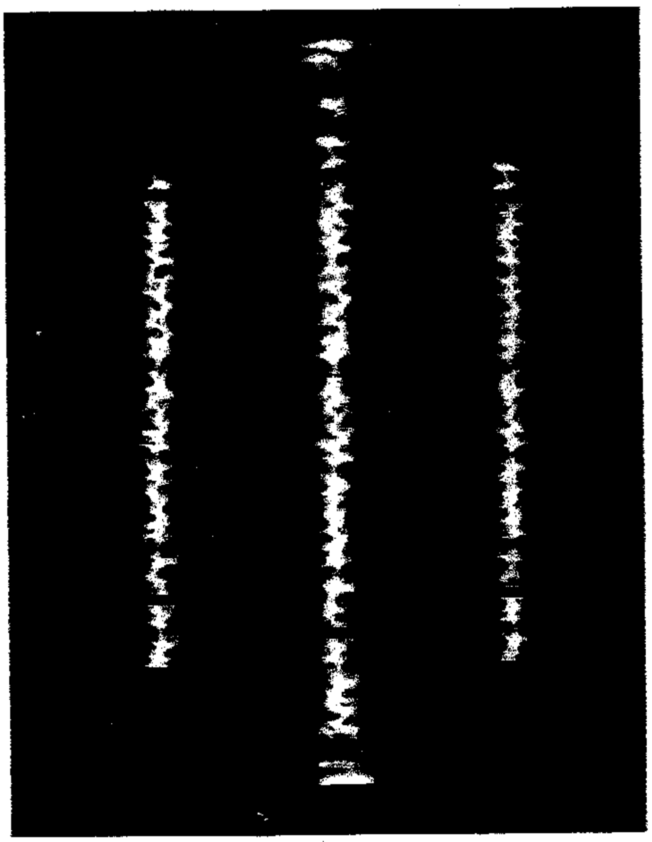
$$\mathbf{J} \times \mathbf{B} = \mathbf{Q}(\mathbf{u})$$

By eliminating **H** from the momentum equation in this manner, the nonlinearity occurring is then of the same quadratic type as in the inertia term, so it may be conjectured that the modifying effect of the presence of conductivity will not change the general character of the turbulent velocity field. Work on isotropic turbulence, mainly by CHANDRASEKHAR (1955) bears out this contention. On the other hand, if one is able to neglect entirely the LORENTZ force in the momentum equation, the turbulent velocity field remains unaffected and the magnetic field can be regarded as a "vector-passive contaminant". The treatment then will follow closely earlier theories developed for a "scalar-passive-contaminant". More details will be given on these considerations today by MOFFATT.

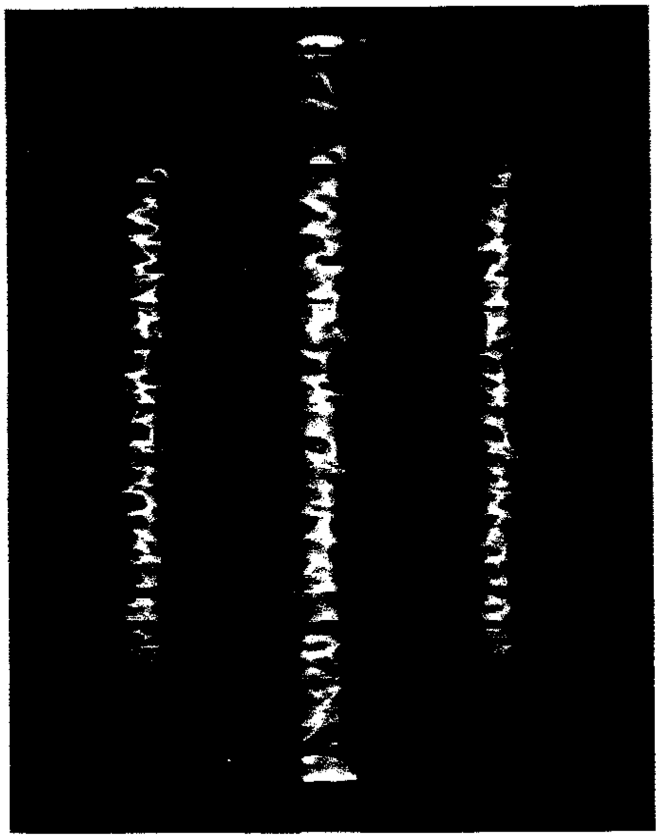
If the turbulent fields are such that the mean velocity gradients as well as the mean magnetic fields (or electric currents) control the phenomenon then, of course, isotropic turbulence is out of the question. Without actually solving the vector equations, the turbulent energy equation REYNOLDS equation) can be developed with the inclusion of electro-magnetic effects. The energy equation may be written either separately for "kinetic" and for "magnetic" turbulent energy or in a combined form for the total turbulent energy (KOVÁSZNAY, 1960). Besides the familiar terms of the turbulent energy equations (generation by mean flow gradients, diffusion by pressure, turbulent diffusion, viscous dissipation) new coupling terms also appear, such as $\overline{\mathbf{J}} \cdot (\overline{\delta \mathbf{u} \times \delta \mathbf{B}})$, the energy transfer from the mean electric current into the «mechanical» turbulence; or $\overline{\mathbf{B}} \cdot (\overline{\delta \mathbf{u} \times \delta \mathbf{J}})$, the transfer from «mechanical» to «magnetic» turbulence by «scrambling» the mean magnetic field by the turbulent motion. A new dissipation term, $\sigma \delta \mathbf{J}^2$, also appears; it is the dissipation of magnetic turbulence by JOULE heat. Turbulent motion of a plasma with no mean flow velocity can be maintained, at least in principle, by feeding energy into the fluctuations by a large D. C. electric current and thus an apparent turbulent electrical resistivity (analogous to the turbulent effect viscosity) is measured in the external electric circuit. There are some experimental suggestions that such a state of affairs may actually exist.



3a



3b



3c

Fig. 3a. — Plasma column (3 views). Cold gas pressure 500μ Hg, working fluid H_2, exposure 1/20μ sec., delay 3. 6μ sec (Courtesy of Dr. E.B. Turner, Aerospace Corp., Los Angeles, California).
Fig. 3b. — As Fig. 3a, except delay 4. 0μ sec.
Fig. 3c. — As fig. 3 a, except delay 4. 4μ sec.

As far as the future is concerned, the next more important step in the development of our understanding of magneto-fluid-dynamic turbulence will be to devise a clear, simple and meaningful experiment, not as much to confirm existing theories as to obtain educated guesses about relative importance of the different approaches.

In the rest of the program, first Dr. M. V. Morkovin will give an account of available experimental evidence on the turbulence found in a supersonic boundary layer. Then Dr. J. Laufer will discuss the sound generation by turbulent shear flows. He will report experimental data obtained in the free stream outside of the supersonic turbulent boundary layer and he will present it against the theoretical background of several existing theories. Finally, Dr. H. K. Moffatt will give a theoretical expose of magneto-fluid dynamic turbulence with special attention to the problem where the magnetic field is regarded as « vector-passive-contaminant ».

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DISCUSSION DE LA COMMUNICATION DU Prof. KOVASZNAY

Secrétaire scientifique : Dr. W. H. REID

Dr. R. Betchov pointed out that, using the same apparatus, the fluctuations of the magnetic field have been recorded. They look turbulent and amount to about 3 % of the mean magnetic field with time scales of about 10^{-7} sec. Correlations and spectra will be measured. Finally, the turbulence depends strongly upon the return path of the electric discharge.

Sir Geoffrey Taylor called attention to the work of Professor Blackett who has found that the instability of a pinch could be correlated with the results obtained from the stability analysis for accelerated interfaces.

Dr. G. K. Batchelor remarked that until a statistically steady state is reached, it would be premature to identify this phenomenon with turbulence. In reply, Professor Kovásznay added that the present experimental conditions prelude the achievement of a statistically steady state. A pinch had been used largely for historical reasons but it did not provide the best experimental conditions; other experiments were now being planned which would overcome this difficulty.

EFFECTS OF COMPRESSIBILITY ON TURBULENT FLOWS

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SOMMAIRE

Les questions sont centrées sur le caractère et le degré de modification dans la structure des champs de vitesse sans divergence (turbulence) entraînée par le couplage des modes acoustique et thermique (entropique).

Le couplage a lieu essentiellement par les variations dans l'espace et dans le temps de la densité, ρ (T, p), de la viscosité μ (T) et de la conductivité thermique k (T), et dépend pour cela des différences de températures moyennes $\Delta \overline{T}$, et nombre de Mach moyen, $\Delta \overline{M}$, maintenues dans ou à travers un champ d'écoulement donné.

Pour de petites valeurs de $\Delta \overline{T}$ et $\Delta \overline{M}$, CHU et KOVASZNAY (1958) analysèrent systématiquement l'interaction entre les modes (d'ordre de grandeur comparable) au second ordre en amplitudes, et identifièrent les processus physiques correspondants. Avec des restrictions plus sévères, MOYAL (1951) étudia l'aspect spectral des interactions dans des champs homogènes. L'idée des modes fournit une base ferme pour les études expérimentales des écoulements supersoniques non stationnaires, à l'aide d'anémomètres à fils chauds (KOVASZNAY, 1953). Une extension des procédés aux domaines subsoniques révéla les effets substantiels de la compressibilité sur les techniques de mesure à des nombres de Mach relativement faibles (MORKOVIN, 1956).

La plupart des études sur les écoulements cisaillés turbulents compressibles (où des effets importants pourraient être produits par de grandes valeurs de $\Delta \overline{M}$ ou $\Delta \overline{T}$ et où la dissipation visqueuse ne peut être négligée) ont été motivées par l'intérêt technologique des paramètres importants de frottement aux parois, et de transfert de chaleur.

Sans clarifier les bases physiques, les diverses généralisations formelles des approximations de PRANDTL, Von KARMAN, BOUSSINESQ, REYNOLDS, etc., ont conduit à des résultats largement différents (voir par exemple Fig. B 12*b* de SCHUBAUER et TCHEN, 1959, où une discussion étendue et la bibliographie de 1956 sont encore présentées).

Les vagues notions sousjacentes de correspondance entre les couches cisaillées compressibles et incompressibles suggérèrent à MAGER (1958) de proposer une planification définie entre ces domaines.

Une tentative de fondement de la théorie de la couche limite sur les conceptions plus modernes de spectres et de similitude en turbulence fut réalisée par LIN et SHEN (1951). Dans un large traité, COLES (1953) concentra l'attention sur la nature asymptotique de l'« unique idéale » couche turbulente (qui est rarement réalisée dans les expériences à grande vitesse, d'où une source réelle de contradictions dans les mesures) et proposa une généralisation aux couches adiabatiques de la loi à la paroi. Les extensions les plus raffinées des conceptions de double similitude des couches en présence de transferts de chaleur à haute vitesse sont probablement celles de ROTTA (1959-1960). Les concordances avec l'expérience qu'il obtient ainsi que d'autres auteurs, sont entravées par les désaccords et les erreurs probables dans des mesures difficiles (surtout celles des températures d'arrêt au voisinage des parois).

Dans un mémoire terminé une semaine avant le Colloque, COLES (1961) présente un

aperçu substantiel et rationnel de la correspondance compressible-incompressible. Peu de traités publiés peuvent supporter les nouveaux critères de concordance entre les points de vues théorique et empirique. En fait, la généralisation antérieure (1953) de la loi des parois de COLES est remplacée.

Du fait de leur complexité (prohibitive dans les régions subsoniques des écoulements supersoniques) les mesures des caractéristiques des fluctuations dans la couche limite ($\overline{u'^2}$, $\overline{\rho'^2}$, $\overline{\rho' \mu'}$) ont été entreprises seulement pour des conditions adiabatiques à la paroi. (KOVASZNAY en 1953, MORKOVIN en 1955-1956); (comprenant les spectres partiels, et la diffraction du mode acoustique); KISTLER (1959) (comprenant le nombre de Mach le plus élevé de 4,67); MORKOVIN et PHINNEY (1958) (comprenant des mesures limitées des corrélations $\overline{u'v'}$ et $\overline{\rho'v'}$).

Une interprétation critique de l'information (MORKOVIN, 1960) a donné une base plus solide à la « similitude ». Même en présence de la dissipation, le mouvement à plus grande échelle serait couplé statistiquement au domaine thermique, presque exclusivement par l'intermédiaire des valeurs moyenne $\overline{\rho}$, $\overline{\mu}$, $\overline{k}$ et la loi à la paroi généralisée de sorte que, avec un facteur d'élargissement latéral variable, il puisse ressembler au mouvement incompressible. Lorsque la vitesse des plus grands tourbillons du courant libre devient sonique et supersonique à des nombres de Mach supérieurs à 4 ou 5, une telle similitude peut produire des déviations croissantes. Les traits saillants du tableau qui se dégage des couches limites turbulentes non hypersoniques sera alors présenté.

SUMMARY

The questions center on the character and degree of change in the structure of divergence-free velocity fields (turbulence) brought about by coupling with sound and thermal (entropy) fields (modes). The coupling occurs primarily through spatial and timewise variation of density ρ (T, p), viscosity μ (T), and heat conductivity k (T), and depends therefore on the " driving " differences in mean temperature, $\Delta\overline{T}$, and mean Mach number, $\Delta\overline{M}$, maintained within or across a given flow field.

For small $\Delta\overline{M}$ and $\Delta\overline{T}$, CHU and KOVASZNAY (1958) analyzed systematically the interactions between the modes (of " comparable magnitudes ") to second order in amplitudes and identified the corresponding physical processes. Under more severe restrictions, MOYAL (1951) studied the spectral aspects of the interactions in homogeneous fields. The concept of modes provided a firm basis for experimental studies of unsteady supersonic flows with hot-wire anemometers (KOVASZNAY, 1953). An extension of the procedures to subsonic fields disclosed substantial compressibility effects on the measuring techniques at relatively low Mach numbers (MORKOVIN, 1956).

Most studies of compressible turbulent shear flows (where large effects could be induced by large $\Delta\overline{M}$ or $\Delta\overline{T}$ and where viscous dissipation cannot be neglected) have been motivated by technological interest in the gross parameters of wall friction and heat-transfer rate. Without clarifying the physical basis, the various formal generalizations of the approaches of PRANDTL, VON KARMAN, BOUSSINESQ, REYNOLDS, etc., have led to widely different results (see, for instance, Fig. B12*b* of SCHUBAUER and TCHEN, 1959), where a comprehensive discussion and the bibliography through 1956 are also found). The underlying vague notions of correspondence between the compressible and incompressible shear layers prompted MAGER (1958) to propose a definite " mapping " between the fields.

An attempt to base the boundary layer theory on the more modern spectral and similarity concepts of turbulence was made by LIN and SHEN (1951). In a comprehensive treatment, COLES (1953) focused attention on the asymptotic nature of the " unique ideal " turbulent layer (which is seldom achieved in high-speed experiments, hence a real source for discrepancies in measurements) and proposed a generalization of the law of the wall-to-adiabatic layers. The most refined extensions of the double-layer similarity concepts in the presence of high-speed heat transfer probably are those of ROTTA (1959, 1960). His and other authors' correlations with experiments are hampered by discrepancies and probable errors in difficult measurements (especially those of stagnation temperature near walls). In a memoir completed a week before the Colloquium, COLES (1961) presents a consistent and rational view of the incompressible-compressible correspondence. Few of the published treatments can pass the new criteria of theoretico-empirical consistency. In fact, COLES' own earlier (1953) generalization of the law of the wall is replaced.

Due to their complexity (prohibitive in subsonic regions of supersonic flows), measurements of fluctuating boundary-layer characteristics ($\overline{u'^2}, \overline{\rho'^2}$, and $\overline{\rho' u'}$) have been carried out only for adiabatic wall conditions (KOVASZNAY (1953); MORKOVIN (1955), (1956) (including partial spectra and sound mode diffraction); KISTLER (1959) (including the highest Mach number of 4.67); MORKOVIN and PHINNEY (1958) (including limited $\overline{u' v'}$ and $\overline{\rho' v'}$ correlations)).

A critical interpretation of the information (MORKOVIN, 1960) provided a firmer basis for " similarity ". Even in presence of dissipation, the larger-scale motion should be statistically coupled to the thermal field almost exclusively through mean values of $\overline{\rho}$, $\overline{\mu}$, $\overline{k}$, and the generalized law of the wall so that with a variable lateral stretching factor, it may resemble the incompressible motion. As the speed of the larger eddies relative to the free stream becomes sonic and supersonic at Mach numbers above 4 or 5, increasing departures from such similarity may occur. The salient features of the emerging picture of the nonhypersonic turbulent boundary layers will now be presented.

Passivity of Compressible Turbulence

Compressibility effects could be expected in presence of large relative velocities of neighboring fluid lumps, of rapid variation in the density ρ, and of consequent dynamic coupling between « modes ». However, making the fluid compressible does not appear to add any substantial sources of vorticity; in particular, the Bjerkness mechanism mentioned by Dr. KOVASZNAY, appears to be weak. A compressible turbulent system then remains essentially passive and large relative velocities disappear in absence of frequent or steady inputs. In fact, even in boundary layers and mixing regions where a steady input is present, we do not observe local *rms* fluctuating Mach numbers above 0.15 ~ 0.2 for free stream Mach numbers, M_e, up to 5.

Dominance of Large-Scale Structure

In order to appreciate the manner in which rapid density variations may effect the dynamics of boundary layers, consider the typical spectra in Fig. 1. These were taken at $\frac{y}{\delta} = 0.2$, in a turbulent boundary layer with thickness $\delta = 0.6$ inch and $M_e = 1.77$. The arrow indicates the frequency which would correspond to eddies of size δ transported at the speed of 0.8 U_e (referred to as magic by previous speakers). Clearly, the dynamics of supersonic boundary layers are governed by large-scale eddies, as at low speeds. To emphasize this aspect, the low-speed spectral shape of velocity fluctuations of KLEBANOFF-DIEHL (1952), taken at the same $\frac{y}{\delta}$, was transformed to the present frequencies by assuming integral scales $L_{xT} = 0.27$ inch $= 0.45\,\delta$ and $L_{x\sigma} = 0.15$ inch $= 0.25\,\delta$. These lengths are to be compared with the low-speed value of 0.4 δ.

The shadowgraph shown yesterday by Dr. ROTTA displayed this large-scale structure optically. Incidentally, both the optical and hot-wire observations at supersonic speeds exhibit the intermittancy phenomena at the free-stream edge, which were emphasized yesterday by Dr. LIEPMANN in low-speed flows.

While no spectra of $\overline{\Delta u \Delta v}$ and $\overline{\Delta T \Delta v}$ have been obtained, there is little doubt that the momentum and heat transfer mechanisms are also dominated by the large-scale

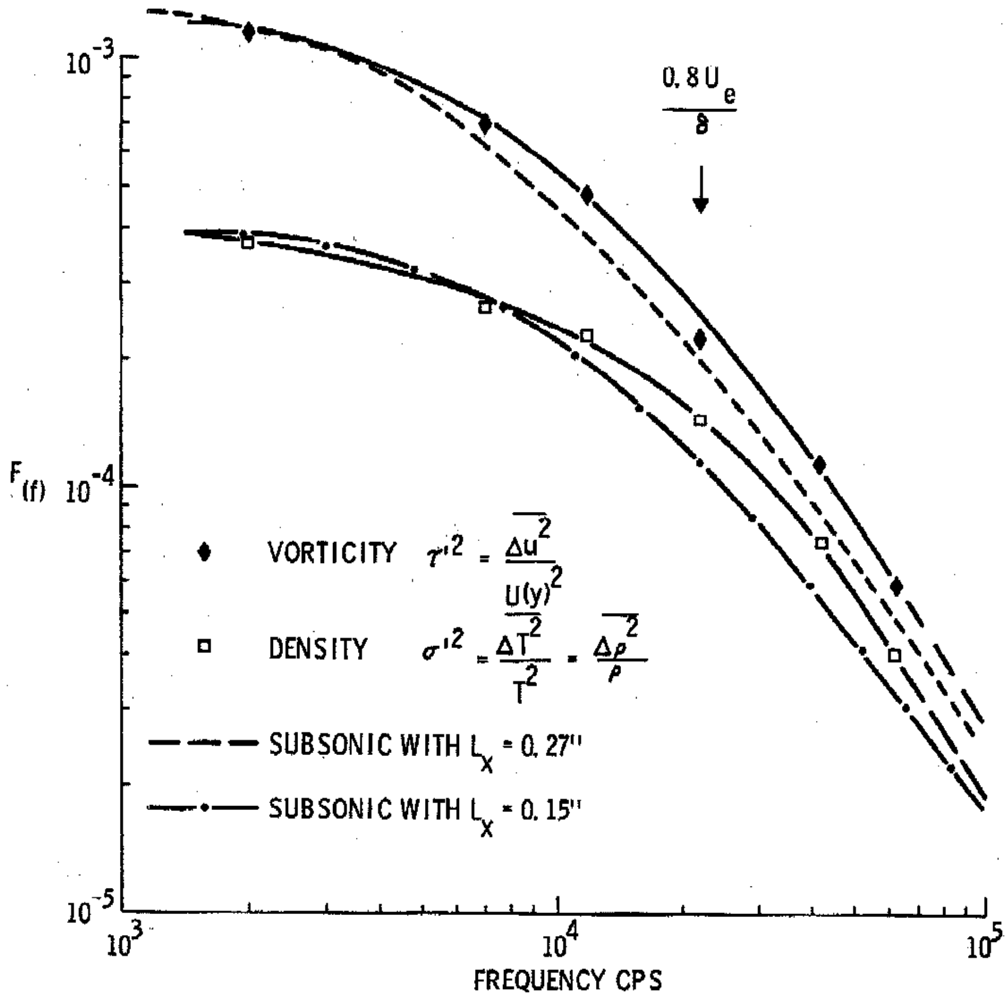


FIGURE 1
Spectra of vorticity and density (entropy)
Fluctuations at $y/\delta = 0.2$, $U_e = 15{,}600$ in./sec, $M_e = 1.77$.

eddies, as at low speeds. At large Reynolds numbers, the scales at which the peculiarly compressible coupling between the velocity field and the thermal field manifests itself, i.e., the scales at which dissipation converts kinetic energy into degraded thermal energy, are thus far removed from the scales at which the essential dynamics of the boundary layer evolve. In the outer layers, we need not seek therefore any direct coupling between Δu^2 and ΔT, which is especially troublesome in the spectral representation. Rather, we can focus our attention on the mechanical energy of the large-scale velocity field and view it as an open system receiving energy from the mean flow and losing energy at the rate $-\bar{\rho}\,\overline{\Delta u \Delta v}\,\frac{\partial U}{\partial y}$ (as will be shown) without need for specification of the small-scale structure * and interactions.

* Frequency requirements are likely to prevent accurate experimental studies of this small-scale structure anyway.

Limited Vorticity-Entropy Coupling in Supersonic Boundary Layers

Thus, we are led to the schematic view of the coupling between the vorticity and entropy modes in a turbulent boundary layer represented in Fig. 2. The paddle wheels emphasize that the turbulent thermal exchanges are driven by the velocity field as at low speeds. The fine structure is outside of these systems receiving the energy $-\rho\,\overline{\Delta u \Delta v}\,\dfrac{\partial U}{\partial y}$ and converting it into degraded thermal energy through the action of μ and k. This constitutes a « compressible » feedback mechanism which is unlikely to affect the essential dynamics of the boundary layer but rather modulates them through (stratified) mean values of $\bar{\rho}(y)$ and $\overline{T}(y)$. In any experimental comparison between low-speed and supersonic measurements and in corresponding theoretical « mappings », an appropriate lateral scaling is therefore in order. In the wall layer the Dorodnitsyn-Howarth scaling appears indicated, but the experimental and theoretical evidence for its usage in the outer layer is not conclusive (Rotta, 1960; Spence, 1960; Coles, 1961).

We note that in contrast to the low-speed case, heat transfer at the wall also modulates the boundary-layer processes by changing the mean values of $\bar{\rho}$ and $\overline{T}$ (see Fig. 2, conduction role). Thus, yet another parameter arises for the specification of the compressible mean velocity field.

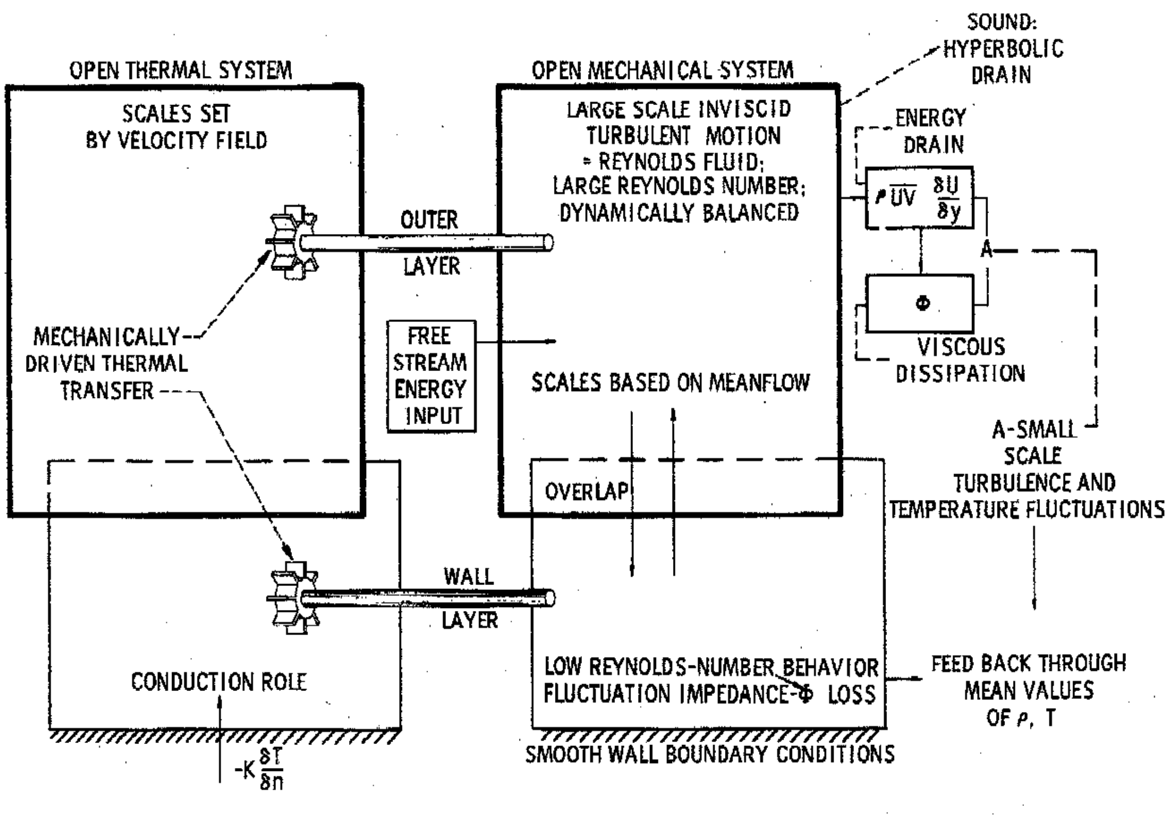
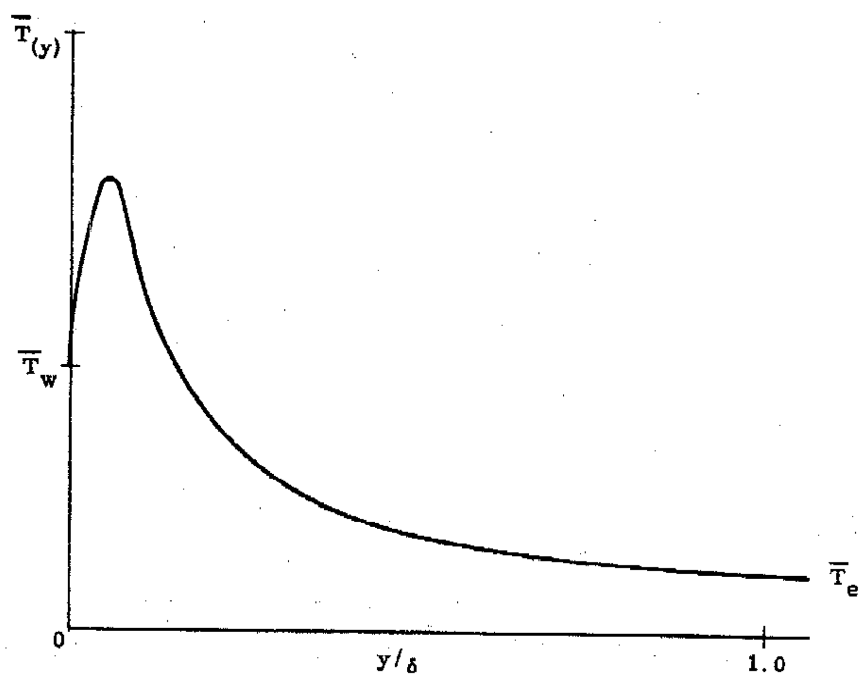


Figure 2
Mean flow features.

Dissipation and Low-Reynolds Number Effects

As at low speeds, most of the dissipation occurs in the wall layer. However, at high speeds, this rather localized heat source has a most important consequence : the local values of $\frac{\bar{\rho}}{\bar{\mu}}$ and therefore the local Reynolds number, Re_y, fall rapidly with increase in speed. For example, the ratio of Re_y at the free-stream Mach number of 5 to the low-speed Re_y at the same small value of $\frac{y}{\delta}$ is on the order of $\frac{1}{20}$ for an insulated wall when the Reynolds numbers of the two layers based on free-stream conditions and δ are the same. The low-Reynolds number behavior at the wall encroaches increasingly upon the main body of the boundary layer with increasing M_e, and may well cause a departure from the normal asymptotic two-layer behavior. (Speculations about $M_e \to \infty$, would undoubtedly require $R_{e_\delta} \to \infty$, or infinite cooling at the wall; at any rate, the results would depend upon the specific limiting process.)



The heat sources in the wall layer grow with M_e, and easily « overpower » effects of cooling at the wall, leading to static temperature profiles with a sharp maximum as in Sketch 1. A measure of both the sharpness of the gradients near the wall as well as of the experimental difficulties, can be found in the fact that no successful measurements of the actual maximum near a cold wall have been reported to date. The prospects of measuring fluctuating quantities in this neighborhood are almost nil.

For the sake of completeness, Fig. 2 also displays the energy loss through radiated sound, which is small in nonhypersonic boundary layer as Dr. Laufer will undoubtedly tell us later.

Reynolds Mechanism in Compressible Flows

The schematic view of Fig. 2 is also supported by a consistent representation of the mean turbulent compressible flow-field as a Reynolds fluid with stresses $-\bar{\rho}\,\overline{\Delta u \Delta v}$, mechanical energy loss $-\bar{\rho}\,\overline{\Delta u \Delta v}\,\frac{\partial U}{\partial y}$, total enthalpy transfer $-\bar{\rho}\,\overline{\Delta H \Delta v}$, etc., as will now be shown. In order to separate the various transport properties from spurious contributions due to mass transport, we consider transport accross mean streamlines, i. e., with zero mass transport*. The mean streamlines in a compressible flow have the direction (in the Δ and in the alternate condensed notation).

$$\tan\theta = \frac{\bar{\rho}\bar{v} + \overline{\Delta\rho\Delta v}}{\bar{\rho}U + \overline{\Delta\rho\Delta u}} = \frac{\bar{\rho}U_2 + \overline{\rho' u_2}}{\bar{\rho}U_1 + \overline{\rho' u_1}} \tag{1}$$

The two terms in the numerator are of the same order in many flows, while the second term in the denominator is of second order with respect to the first and can be neglected; the overall approximation of a compressible Reynolds fluid being no better than second order (e. g. in the mean equation of state of the gas). The material derivative along the mean streamline will be denoted as $\overline{\rho\frac{D}{Dt}}$, or in the condensed summation notation

$$(\bar{\rho}U_j + \overline{\rho' u_j})\,\frac{\partial}{\partial x_j} \equiv \overline{\rho\frac{D}{Dt}} \tag{2}$$

The averaged compressible Navier-Stokes equations then yield the Reynolds momentum equations in the form :

$$\overline{\rho\frac{D}{Dt}}U_i = -\frac{\partial\bar{p}}{\partial x_i} + \frac{\partial\,(\bar{\tau}_{ij} - \bar{\rho}\,\overline{u_i u_j})}{\partial x_j} \tag{3}$$

In this formulation, the turbulent stress tensor $-\bar{\rho}\,\overline{u_i u_j}$ takes the same form as for incompressible flows, playing the same role as the laminar stress $\bar{\tau}_{ij}$.

In order to check the consistency of the equation for the kinetic energy of the mean flow, we must first inquire as to its proper form and meaning for viscous compressible flows in general. Taking a scalar product of the (unaveraged) velocity $\mathfrak{U}_i$ with the Navier-Stokes equations, we obtain, after some rearranging :

$$\frac{\partial}{\partial x_j}\left(-\mathfrak{U}_i p\,\delta_{ij} + \mathfrak{U}_i \tau_{ij}\right) = \frac{\rho}{2}\frac{D\mathfrak{U}_i\mathfrak{U}_i}{Dt} + \left\{\tau_{ij}\frac{\partial\mathfrak{U}_i}{\partial x_j}\right\} + \left[-p\frac{\partial\mathfrak{U}_i}{\partial x_i}\right] \tag{4}$$

The left side represents the instantaneous rate of the total mechanical work being done on an element of fluid as it moves along is path. For incompressible fluids, this input is either converted into mechanical energy of motion or lost to the mechanical system through the dissipation term in braces, usually designated as Φ. The additional term in the bracket represents the rate of change of the mechanically recoverable internal energy of the gas generated by compression-expansion along the path.

* The mean continuity equation is then automatically satisfied, and in the form

$$\overline{\rho\frac{D}{Dt}}\left(\frac{1}{\rho}\right) = \frac{\partial U_j}{\partial x_j}$$

exhibits only the mean compressibility effects.

To obtain the mechanical energy equation of the mean motion for turbulent compressible flows, we proceed similarly by taking the scalar product of U_i with the Reynolds equations and rearranging terms as in (4)

$$\frac{\partial}{\partial x_j}(-U_i\bar{p}\,\delta_{ij}+U_i\,\overline{\tau_{ij}}-U_i\,\bar{\rho}\,\overline{u_i u_j})=\frac{\bar{\rho}}{2}\frac{\overline{D}U_i U_i}{\partial t}+\left\{(\overline{\tau_{ij}}-\bar{\rho}\,\overline{u_i u_j})\frac{\partial U_i}{\partial x_j}\right\}+\left[-\bar{p}\frac{\partial U_i}{\partial x_i}\right] \tag{5}$$

The interpretation of Eq. (5) is analogous to that of Eq. (4) : the net rate of work done on a fluid element by mean motion is converted into the change of its mean kinetic energy along its mean path, lost through dissipation and turbulence production $-\bar{\rho}\,\overline{u_i u_j}\frac{\partial U_i}{\partial x_j}$ and partially stored as internal energy of the element. Thus, the representation of the mean compressible turbulent flows by means of additional turbulent stresses $-\bar{\rho}\,\overline{u_i u_j}$, acting on fluid elements along mean streamlines, is seen to be consistent when both the momentum and mechanical energy are considered.

When the combined mechanical and thermal energies of a fluid element are observed* along the mean streamline, a minor modification of YOUNG's analysis (1951) leads to a consistent view in terms of stagnation or total enthalpy $\overline{H}$, $\overline{H}=\bar{h}+\frac{U_i U_i}{2}+\frac{\overline{u_i u_i}}{2}$ $=C_p\overline{T}_t$ (for perfect gas) as well. The energy equation becomes

$$\bar{\rho}\frac{\overline{D}\overline{H}}{Dt}=\frac{\partial}{\partial x_j}\left[\frac{\bar{k}}{C_p}\frac{\partial\overline{H}}{\partial x_j}-\bar{\rho}\,\overline{u_j H'}+\bar{\mu}\left(1-\frac{1}{P_r}\right)\frac{\partial}{\partial x_j}\left(\frac{U_i U_i}{2}+\frac{\overline{u_i u_i}}{2}\right)\right] \tag{6}$$

and shows that the turbulent energy-transfer vector, $-\bar{\rho}\,\overline{u_j H'}$, has also the same structure as at low speeds.

While we have Eqs (3) and (6) before us, we note with A. D. YOUNG (1951), that when the pressure gradient term in Eq. (3) and the laminar $\bar{\mu}$ term* in Eq. (6) can be neglected, the equations admit the solution

$$C_p\overline{T}_t-C_p\overline{T}_{tw}=P_{r_w}\frac{\overline{q_w}}{\overline{\tau_w}}U \tag{7}$$

$$C_p\Delta T_t(t)\quad=P_{r_w}\frac{\overline{q_w}}{\overline{\tau_w}}\Delta u(t) \tag{8}$$

where $\overline{q_w}$, the heat-transfer rate at the wall, enters through the boundary conditions. This solution, henceforth called the Strong Reynolds Analogy (S. R. A.), will be discussed later.

Turbulent Energy in Compressible Boundary Layers

From the preceding discussion we would expect that dimensionless shear stress $\bar{\rho}(y)\frac{\overline{\Delta u\Delta v}}{\overline{\tau_w}}$ and energy of longitudinal fluctuations $\bar{\rho}(y)\frac{\overline{\Delta u^2}}{\overline{\tau_w}}=\frac{\bar{\rho}(y)}{\bar{\rho}_w}\frac{\overline{\Delta u^2}}{U_\tau^2}$ would

* In this case the losses, $\Phi-\bar{\rho}\,\overline{u_i u_j}\,\partial U_i/\partial x_j$, in the mechanical energy are converted into a gain in thermal energy so that these terms disappear from Eq. (5). The question of how locally and rapidly this conversion may occur is connected with the question of existence of an energy (Crocco) integral.

* Such as when $Pr=1$ or when turbulent processes dominate.

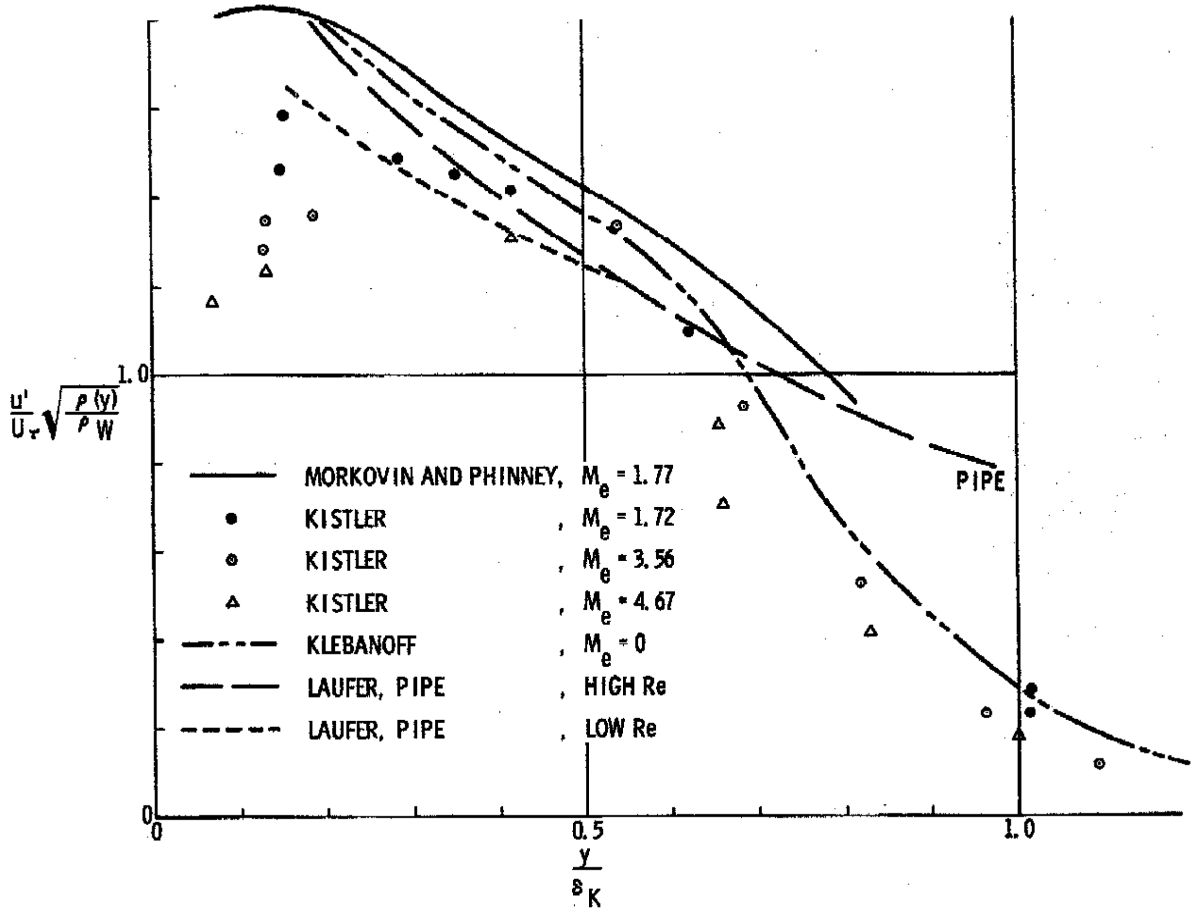


FIGURE 3
Profiles of dimensionless local velocity fluctuations.

depend little on Mach number, if an appropriate lateral scaling and the influence of the low Reynolds numbers at the wall are taken into account. (Here $\overline{\tau_w}$ is the mean shear at the wall, and $U_\tau^2 = \frac{\overline{\tau_w}}{\overline{\rho_w}}$, as usual). Very limited measurements of the above shear stress indicate agreement with low speed within 15 to 25 %. The longitudinal velocity fluctuations, displayed in Fig. 3, appear to scale best with the ratio of the undistorted lateral length y to the boundary layer thickness δ_K. Laufer's « incompressible » measurements in pipes (LAUFER, 1955) with a ratio of Reynolds numbers of 0.1 indicate the probable effect of low wall Reynolds number on fluctuations plotted against $\frac{y}{\delta_K}$. The gradual decrease in the dimensionless fluctuations with increase in Mach number, i. e., with decrease in wall Reynolds number under the present adiabatic conditions, reflects the same trend but may also be influenced by a systematic effect of the decreasing Reynolds number on the measuring instrument. Since high-speed measurements are very difficult*, the agreement with the overall picture is indeed gratifying.

* For example, there is little doubt that in the outer one-third of the boundary layer, the Morkowin-Phinney measurements are inferior to those of Kistler who had considerably less background noise.

As already mentioned, the experimental difficulties will undoubtedly preclude any substantial clarification of the decay and transfer of the above turbulent energy. Yet, the thermodynamic mean energy equation [Eq. (6)] does contain information on the combined effects of energy production, dissipation, diffusion and gradient transfer. For adiabatic wall conditions, the corresponding experimental results, such as that of Fig. 4, show that the stagnation or total temperature, $\overline{T}_t$, is remarkably constant in regions of high Reynolds number. In general, approximately half of the drop of $\overline{T}_t$ to the wall recovery temperature occurs inside the wall layer $\bar{\rho} y \sqrt{\frac{\tau_w}{\bar{\rho}}} \leqslant 60$, even through the slope $\left(\frac{\partial \overline{T}_t}{\partial y}\right)_w$ is zero at an adiabatic wall. A temperature deficiency at the wall would be expected for gases with Prandtl number less than unity in regions where laminar transport is important.

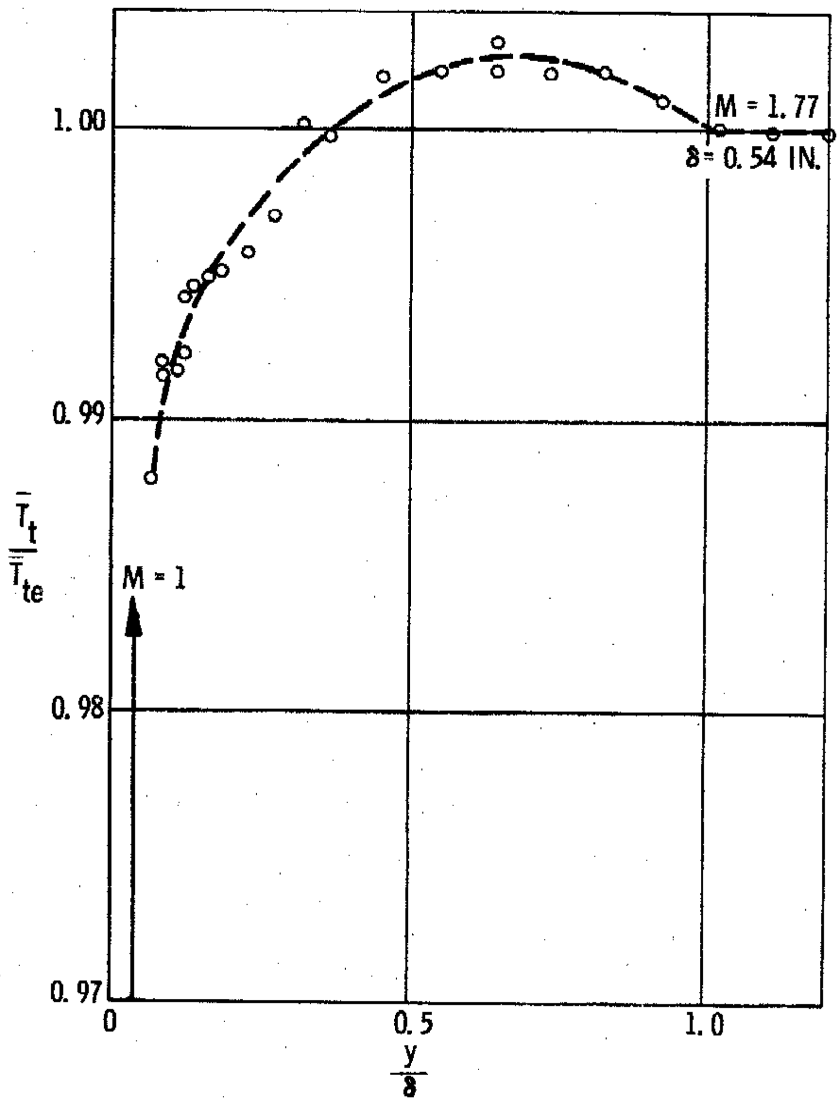


FIGURE 4
Variation of mean total temperature, T_t, AT $M_e = 1.77$.

Thermal Energy and Temperature Fluctuations

Associated with the behavior just described are a number of questions which have not been fully explored and understood, e. g., the empirical constancy of the recovery factor with Mach number, the existence and the form of a Crocco energy equation, the proper generalization of the law of the wall, the Reynolds analogy at the wall (especially in presence of roughness or pressure gradients), etc. As a matter of fact, once we have glimpsed the reassuring picture that the essential mechanisms of the nonhypersonic turbulent boundary layers differ little from those at low Mach numbers, the interesting features which can teach us something new about turbulent fields are those dealing with the relationship between the temperature and velocity fluctuations. For instance, as a companion to Fig. 1, the measured correlation coefficient

$R_{\sigma\tau} = \dfrac{\overline{\Delta T \Delta u}}{\Delta T' \Delta u'}$, and the ratio $\dfrac{\sigma'}{\tau'} = \dfrac{\dfrac{\Delta T'}{\overline{T}(y)}}{\dfrac{\Delta u'}{\overline{U}(y)}}$ are displayed as functions of frequency in Fig. 5. The temperature and horizontal velocity fluctuations of the large eddies are

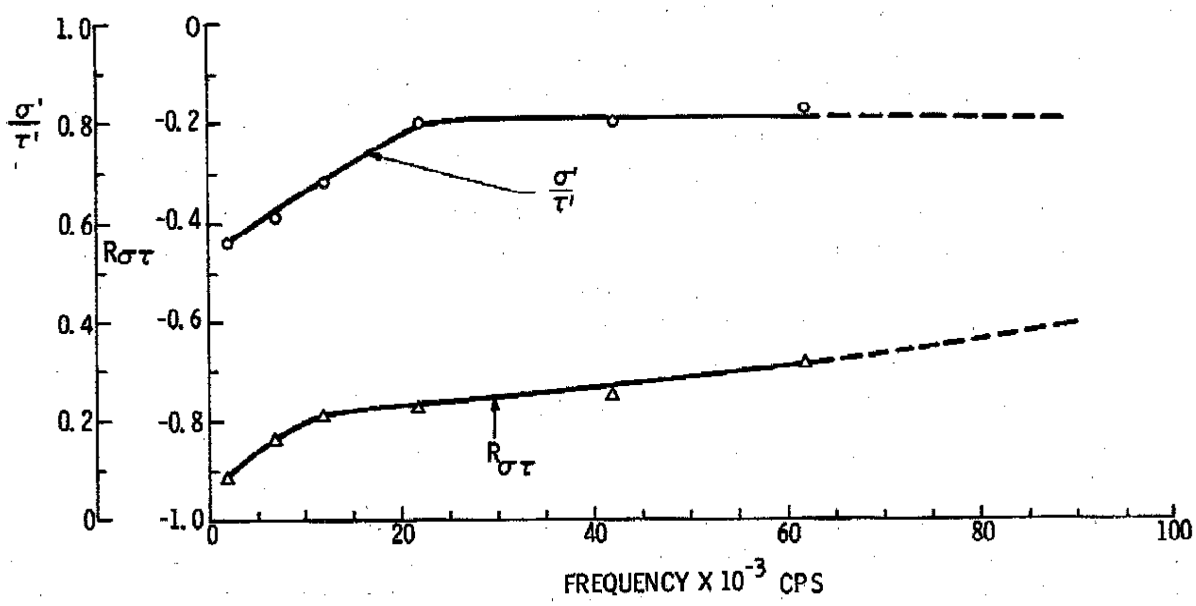


FIGURE 5
Relations between vorticity and density (entropy) fluctuations
Across the spectrum : Y = 0.12 in., M(y) = 1.33, M_e = 1.77.

highly anticorrelated « as if a lump of fluid which is warmer than the average came from a layer closer to the wall, with a velocity less than the average » [KOVASZNAY (1953)]. In fact, for adiabatic flows ($\overline{q}_w = 0$), the Strong Reynolds Analogy [Eq. (8)] leads to

$$\Delta T_t(t) = 0 \tag{9}$$

and its linearized version to

$$\Delta T_t(t) = \Delta T(t) + \frac{U(y)}{C_p} \Delta u(t) = 0 \; ; \tag{10a}$$

$$\frac{\sigma'}{\tau'} = \frac{\dfrac{\Delta T'}{\overline{T}(y)}}{\dfrac{\Delta u'}{\overline{U}(y)}} = (\gamma - 1)\, M^2(y) \; ; \tag{10b}$$

$$R_{\sigma\tau} = \frac{\overline{\Delta T \Delta u}}{\Delta T' \Delta u'} = -1 \; ; \tag{10c}$$

$$\frac{\Delta T'}{\overline{T}_w - \overline{T}_e} = 2 \frac{U(y)}{U_e} \frac{\Delta u'}{U_e} \; ; \tag{10d}$$

$$Pr_{\text{turb}} = \frac{\overline{\Delta v \Delta u}}{\overline{\Delta v \Delta T}} \frac{\partial \overline{T}(y)}{\partial U(y)} = +1 \, . \tag{10e}$$

The experimental value of 0.6 for $\frac{\sigma'}{\tau'}$ of the largest eddies as against 0.7 of Eq. (10*b*) and the less-than-perfect anticorrelation $R_{\sigma\tau}$ in Fig. 5, represent a measure of the departure from S. R. A. for the large-scale motion. As could be expected, the anti-correlation decreases as eddies become smaller. The explanation for the rise of $\frac{\sigma'}{\tau'}$ with frequency probably lies in the difference in behavior between scalar and vector fields. It would be interesting to obtain similar information in terms of eddy structure for rough-wall conditions, for which the variations in the Reynolds analogy at the wall hint at an effective turbulent Prandtl number larger than unity.

The experimental evidence relative to conclusions (9) and (10) is as follows.

The correlation coefficient $R_{\sigma\tau}$, [Eq. (10*c*)], varies little with M_e or y for a given M_e and for a given instrument, and ranges * from —0.7 (especially for Platinum-Rhodium hot wires without sleeves at supports) to —0.9 (especially for Tungsten hot wires with protective sleeves at supports).

The relations (10*b*) and (10*d*) are verified within 20 % or less.

Limited measurements of turbulent Pr, [Eq. (10*e*)], yielded results between 0.9 and 0.93.

However, the key consequence of S. R. A. in adiabatic flows, Eq. (9), was not born out, as can well be seen from Fig. 6 **. This undoubtedly stems from the unrealistic demand that the outer-flow solution, Eqs (7) and (8), appropriate at high Reynolds numbers, extend all the way to the wall. Apparently, a midler form of S. R. A., resting on the negligibility of the dimensionless lateral transport of the total temperature in the outer layers of adiabatic boundary layers $\frac{\overline{\Delta v \Delta T_t}}{\overline{\Delta v \Delta T}}$, may be sufficient to bring about

* These discrepancies are a measure of the experimental uncertainty with the rather difficult technique. Since the wires have different sensitivities to temperature and velocity fluctuations, the relative smallness of the discrepancies would appear to validate the technique.

** Only the extreme measurements, as noted above, were selected from the Morkovin-Phinney set. There were many more measurements, generally in between the values shown here.

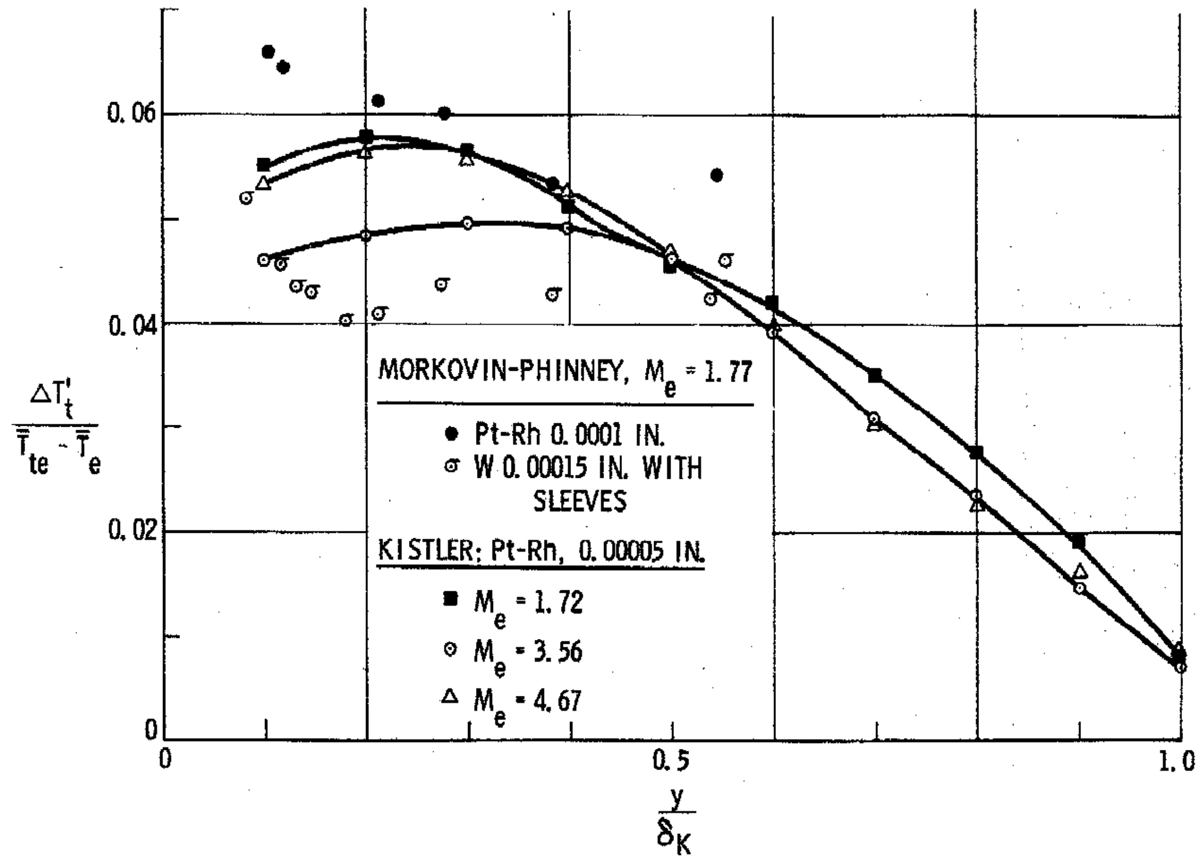


FIGURE 6
RMS fluctuations in total temperature.

the rather satisfactory agreement with the practically useful results of Eqs (10*b*) through (10e). Limited measurements gave values of 0.06 and less for this ratio.

In conclusion, the dimensionless fluctuation profiles of Figs 3 and 6 provide a basis for general assessments of fluctuation levels in adiabatic compressible boundary layers, consistent with the idea of the M-independence of the basic mechanisms. One could recommend careful coupled fluctuation and mean-flow measurements in supersonic turbulent boundary layers with heat transfer and/or roughness, and in supersonic mixing regions. These experiments are feasible with our present techniques, and should throw light on the more interesting questions just listed (some of which can also be studied more accurately at low speeds). However, we can expect with confidence that the essential dynamics of these supersonic shear flows will follow the incompressible pattern.

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SOUND RADIATION FROM A TURBULENT BOUNDARY LAYER

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SOMMAIRE

Si la restriction d'incompressibilité dans le problème de la turbulence est levée, le phénomène de radiation d'énergie sous la forme sonore provenant de la zone turbulente apparaît.

Pour calculer cette énergie rayonnée, il est montré que de nouvelles grandeurs statistiques, comme les tenseurs de corrélations spatio-temporelles, doivent être connues dans la zone turbulente, en plus des grandeurs conventionnelles. Pour le cas particulier de la couche limite turbulente, il y a des indications montrant que l'intensité du rayonnement ne devient notable qu'en écoulements supersoniques. Sous cette condition, le récent travail de PHILLIPS est examiné en même temps que quelques résultats expérimentaux de l'auteur. Il est montré que l'aspect qualitatif du champ rayonné (intensité, directivité) comme prévu par la théorie sont compatibles avec les expériences; cependant, même pour les nombres de Mach les plus élevés, quelques-unes des hypothèses de la théorie asymptotique ne sont pas encore satisfaites dans les expériences.

Finalement la question de la réduction de la turbulence dûe au rayonnement est discutée, avec ce résultat que dans le domaine des nombres de Mach couvert par les expériences, la perte d'énergie de la couche limite dûe au rayonnement est un faible pourcentage du travail accompli par les frottements à la paroi.

SUMMARY

If the restriction of incompressibility in the turbulence problem is relaxed, the phenomenon of energy radiation in the form of sound from the turbulent zone arises. In order to calculate this radiated energy, it is shown that new statistical quantities, such as time-space correlation tensors, have to be known within the turbulent zone in addition to the conventional quantities. For the particular case of the turbulent boundary layer, indications are that the intensity of radiation becomes significant only in supersonic flows. Under these conditions, the recent work of PHILLIPS is examined together with some experimental findings of the author. It is shown that the qualitative features of the radiation field (intensity, directionality) as predicted by the theory are consistent with the measurements; however, even for the highest Mach number flow, some of the assumptions of the asymptotic theory are not yet satisfied in the experiments. Finally, the question of turbulence damping due to radiation is discussed, with the result that in the Mach number range covered by the experiments, the energy lost from the boundary layer due to radiation is a small percentage of the work done by the wall shearing stresses.

Introduction

When the compressibility of the fluid is taken account, a new aspect of the turbulence problem will arise. In a compressible fluid a disturbance from a source will propagate at a finite speed and will influence the flow field *over a finite distance in a given time.* This means that in calculating the flow properties at a given point and time, it will now be necessary to know the behavior of the disturbance source at a certain earlier time. Thus, the concept of retarded time and retarded potential naturally arises. This fact is reflected in the statistical description of a fluctuating flow field; in order to calculate, for instance, the pressure fluctuations emanating from a turbulent shear field, it will now be neccessary to know certain space-time correlation functions within the shear field heretofore not considered.

In order to fix our ideas, we will choose a definite geometry for a turbulent shear flow : the boundary layer. Thus, we have a turbulent fluid streaming over a rigid wall and want to examine the time-dependent pressure field outside of the layer. Within the layer the fluctuations may be described primarily in terms of vorticity and entropy modes and, to a lesser extent, sound modes. Outside the layer the first two modes die out rapidly, so that at a sufficiently large distance from the shear zone (several wavelengths away) one expects to find only fluctuations of the sound mode type present, usually referred to in the literature as aerodynamic noise. We will seek a relation between this sound field and the fluctuations within the boundary layer.

The mathematical tools to handle the radiation have been well developed in the electromagnetic, acoustic and nonstationary supersonic theory. Thus, once the aerodynamic noise problem has been properly formulated and linearized — and this can be done with reasonable assumptions — in principle, at least, a solution can be obtained. The main difficulty and the reason for the rather slow progress in this field is the fact that the solution is written in terms of the aforementioned statistical quantities of the turbulence field about which very little if any information is available.

It is interesting to note that one new feature in a compressible turbulence is the fact that the pressure energy radiated away from the turbulent zone represents a new form of energy loss besides the dissipation. The question naturally arises as to whether or not, at sufficiently high Mach number, the radiation can be intense enough to exceed the rate of turbulence production and thus dampen out the turbulence.

The main purpose of this paper is not to give a comprehensive literature survey on the subject, but rather to extract those ideas that seem to be most helpful in understanding the mechanism of radiation in the case of a turbulent boundary layer. Since recent measurements indicate that the intensity of radiation becomes significant mainly in supersonic flow, theories that depend on the assumption $M \ll 1$ will be merely touched upon; the discussion will concentrate on the supersonic problem.

Formulation of the problem

Taking the divergence of the momentum equation and using the continuity equation, one obtains

$$\frac{\partial^2 \rho}{\partial t^2} - \nabla^2 p = \frac{\partial^2 \rho u_i u_j}{\partial x_i \partial x_j} + \frac{\partial^2 \tau_{ij}}{\partial x_i \partial x_j} \tag{1}$$

where τ_{ij} is the viscous stress tensor.

In order to eliminate the density (at least from the leading terms), Phillips has rewritten this equation in the form (Ref. 1)

$$\frac{D^2}{Dt^2} \log p - \frac{\partial}{\partial x_i} a^2 \frac{\partial}{\partial x_i} \log p = \gamma \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} + \gamma \frac{D}{Dt}\left(\frac{1}{C_p} \frac{DS}{Dt}\right) - \gamma \frac{\partial}{\partial x_i} \frac{1}{\rho} \frac{\partial \tau_{ij}}{\partial x_j} \tag{2}$$

The last two terms in the equation represent entropy fluctuations and viscous effects. If we restrict ourselves to small fluctuations and to regions not too far from the shear layer, where diffusion effects are not important, these terms may be neglected.

The left-hand side of the equation has the form of a wave equation in which the time derivatives have been replaced by those following the motion, and the propagation velocity is a variable. On the right-hand side, the velocity term is usually referred to in the literature as the pressure-generation term. This nomenclature, however, is somewhat misleading and needs some clarification. The velocity fluctuation should of course be considered as an independent variable together with the pressure. This, unfortunately, renders the problem hopelessly complicated. One could adopt the point of view (see, for instance, Ref. 2) that the velocity field within the shear layer is known from measurements, and therefore the right-hand side may be considered a known forcing function for the wave equation. A much more satisfying approach is that of trying, by a suitable assumption, to decouple the pressure field from the velocity field : provided the Mach number is not very high, one may assume that the velocity fluctuation within the shear layer has predominantly vorticity modes; that is to say, the noise field generated by the turbulent shear layer will contribute only a negligible velocity field within the layer. There is some experimental evidence which indicates that this assumption is reasonable. While the pressure fluctuations in the far field vary an order of magnitude in the Mach number range considered, the velocity fluctuation field in the boundary layer (in an appropriately normalized form) does not change as was shown in the previous paper by Morkovin; similarly the wall pressure fluctuations, measured recently by Kistler, vary only slowly with Mach number (see Fig. 1; $\frac{\tilde{p}}{\tau_w}$ is the ratio of rms pressure fluctuation to wall shearing stress)*.

Thus, if the velocity field within the boundary layer is known *a priori*, the problem reduces to solving a wave equation with a known source term. The mathematical difficulty in the solution lies mainly in the fact that the governing partial differential equation has variable coefficients, and some suitable simplification has to be made in order to obtain a solution. In the literature, one finds two approaches which will be discussed below.

a. Low Mach number solution : Lighthill succeeded in reducing the problem to that of classical acoustics by considering flows with $M \ll 1$ (Ref. 3). Under this circumstance,

* The author wishes to express his thanks to Professor Kistler for permission to use these results before publication.

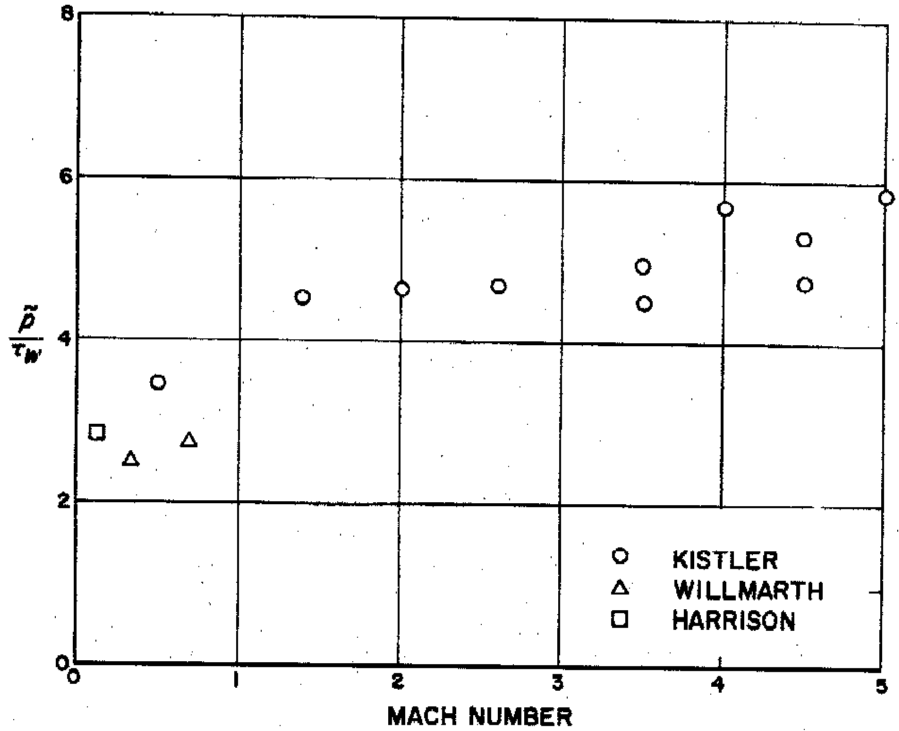

FIGURE 1
Wall pressure fluctuation level.

we can replace the density in the generation term (Eq. 1) by a constant value and neglect a term of the form

$$\frac{1}{a_\infty^2}\frac{\partial^2(a_\infty^2\rho - p)}{\partial t^2} = \frac{1}{a_\infty^2}\left(\frac{a_\infty^2}{\bar{a}^2} - 1\right)\frac{\partial^2 p}{\partial t^2}$$

under the assumptions that the temperature of the flow field is nearly uniform and the fluctuations are locally isentropic ($\bar{a}$ is the local mean speed of sound; subscript ∞ refers to free-stream conditions). With this simplification we obtain

$$\frac{1}{a_\infty^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \rho_\infty \frac{\partial^2 u_i u_j}{\partial x_i \partial x_j}$$

The problem is thus reduced to finding the pressure fluctuations in a uniform acoustic medium at rest produced by certain types of sources. The solution for the case of the boundary layer was first obtained by Curle (Ref. 4) in a quadrature form in which the integrands contain the Reynolds stresses and pressure forces taken at appropriately retarded times. (Other authors, Refs. 5 and 6, gave the results in somewhat different form.) It follows that the pressure intensity $\overline{p^2}$ will contain the space-time correlations of these quantities. In order to obtain numerical values for $\overline{p^2}$, it is, of course, necessary to know the correlations throughout the shear zone. For the case of a boundary layer of constant thickness, Phillips estimated the intensity of pressure radiation and found it to be negligibly small (Ref. 7). Indeed, measurements of $\overline{p^2}$ in supersonic flows indicate a very rapid decrease in intensity as the Mach number approaches subsonic values.

Thus it appears that for boundary layers the radiation problem becomes interesting mainly in supersonic flows. In this case, however, the mathematical difficulties become much larger. Lighthill's very useful acoustical analogy will have to be reexamined. The physical problem, of course, becomes more complicated : the sound velocity within the layer will vary appreciably because of the large temperature gradients; the convection velocities of the sources can no longer be neglected and can be subsonic or supersonic with respect to the free stream.

b. Solution for $M \to \infty$: Phillips has studied this problem using Eq. (2) and has succeeded in obtaining a solution for the asymptotic case of $M \to \infty$ (Ref. 1). Although he considered a free shear layer, his theory can be easily adapted to the boundary layer. The main results may be described as follows : the wave-number frequency energy spectrum of the pressure fluctuations at a point outside the boundary layer corresponding to a given wave number k and frequency n is contributed entirely by a certain critical layer within the boundary layer that lies at a distance Y from the wall where $n + k_1 \mathrm{U}(\mathrm{Y}) = 0$ (k_1 is the x component of the vector k). In other words, one may consider at Y a frozen eddy pattern that is convected downstream with a velocity U (Y) which is supersonic with respect to the free stream (Fig. 2). The pattern thus moves

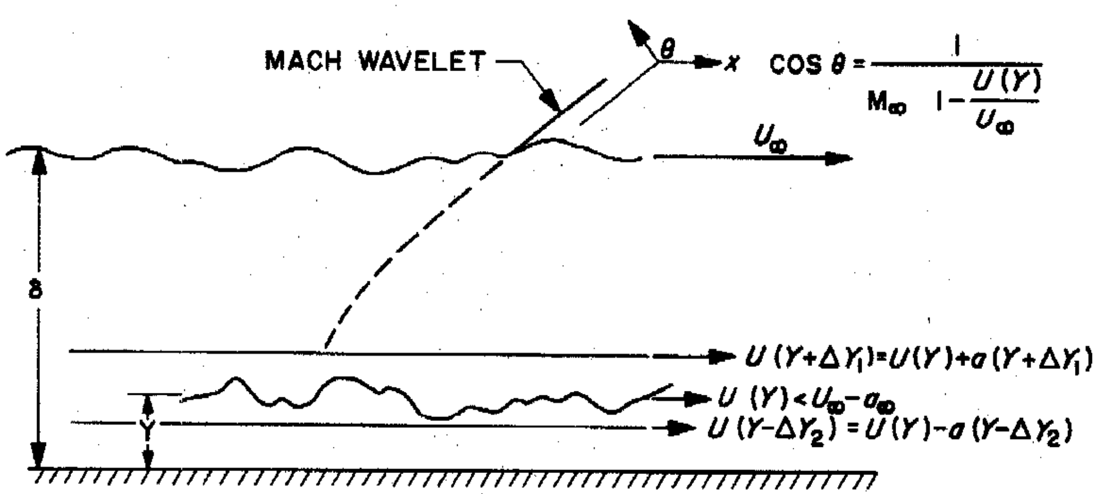

FIGURE 2
Schematic diagram of the radiation mechanism.

like a wavy wall in a supersonic stream and radiates energy in the form of Mach waves, the direction of the waves depending on the relative velocity between U (Y) and the free stream. The resulting pressure spectrum has the form

$$\frac{\pi(\vec{k})}{\delta \bar{p}_\infty^2} \backsim \mathrm{M}_\infty^{3/2} \int_0^\infty \frac{\dfrac{d \dfrac{\mathrm{U}}{\mathrm{U}_\infty}}{d \dfrac{y}{\delta}} k_1^2 \, \psi\left(\dfrac{\mathrm{Y}}{\delta}, \vec{k}, n\right)}{\dfrac{a^2}{a_\infty^2} \left[\mathrm{U}_\infty - \mathrm{U}\right]} \, d\frac{\mathrm{Y}}{\delta}$$

where $n = -k_1 U(Y)$, and $\psi(\frac{Y}{\delta}, \vec{k}, n)$ is the wave-number frequency spectrum of the v' fluctuations in the boundary layer at Y.

Following Phillips, if one now makes the rough approximation that $\psi(\frac{Y}{\delta}, \vec{k}, n) \sim \psi(k)\,\theta(k)$ is independent of Y where $\theta(k)$ is an integral time scale of the v' spectrum of the order of $\frac{\delta}{U_\infty}$, one obtains for the pressure intensity

$$\frac{\overline{p'^2}}{\bar{p}^2} \backsim M_\infty^{3/2} \left[\frac{\partial \frac{v'}{U_\infty}}{\partial \frac{x}{\delta}}\right]^2 \int_0^\infty \frac{\frac{d\frac{U}{U_\infty}}{d\frac{y}{\delta}}\, d\left(\frac{Y}{\delta}\right)}{\frac{a^2}{a_\infty^2}\left[1 - \frac{U}{U_\infty}\right]}$$

The following comments may be made about the result :

1. The pressure field outside of the boundary layer is uniform independent of the distance from the plate.

2. The variation of the radiated pressure intensity with Mach number cannot be expressed explicitly. The value of the integrand decreases rapidly with Mach number (roughly as M^2) so that the intensity is expected to vary much slower than $M^{3/2}$.

3. Within the framework of the theory, directionality of the radiation can also be predicted. It may be shown that as the Mach number of the free stream is increased, a larger portion of the total radiated energy will be concentrated in a given direction, the direction of propagation approaching the perpendicular to the boundary layer as $M \to \infty$.

Discussion

In this case, as in any type of asymptotic solution, one would like to know whether the results of such a solution could be applied to finite values of Mach numbers. In this section we will examine the existing experimental information on sound radiation in the light of Phillips' theory.

In a recent work (Ref. 8) it was shown that in a supersonic wind tunnel the fluctuations in the free stream are mainly pressure or sound waves emanating from the turbulent boundary layers of the four tunnel walls. The sound field intensity was found to be very uniform a few wavelengths outside of the boundary layer. Figure 3 shows the normalized $\overline{v'^2}$ fluctuations near the edge of the boundary layer for several Mach numbers. It is seen that in the free stream the sound field intensity (where v' is proportional to p') is uniform indeed. Furthermore, the nonuniformity in intensity extends farther out of the boundary layer for the low Mach number flow. This is not surprising since, as will be seen later, the nondimensional wave length of the sound field $\frac{\lambda}{\delta} = -\left(\frac{\lambda_x}{\delta}\right)\cos\theta$ is larger at lower Mach numbers. (θ is the angle between the normal to the wave front and the flow direction.)

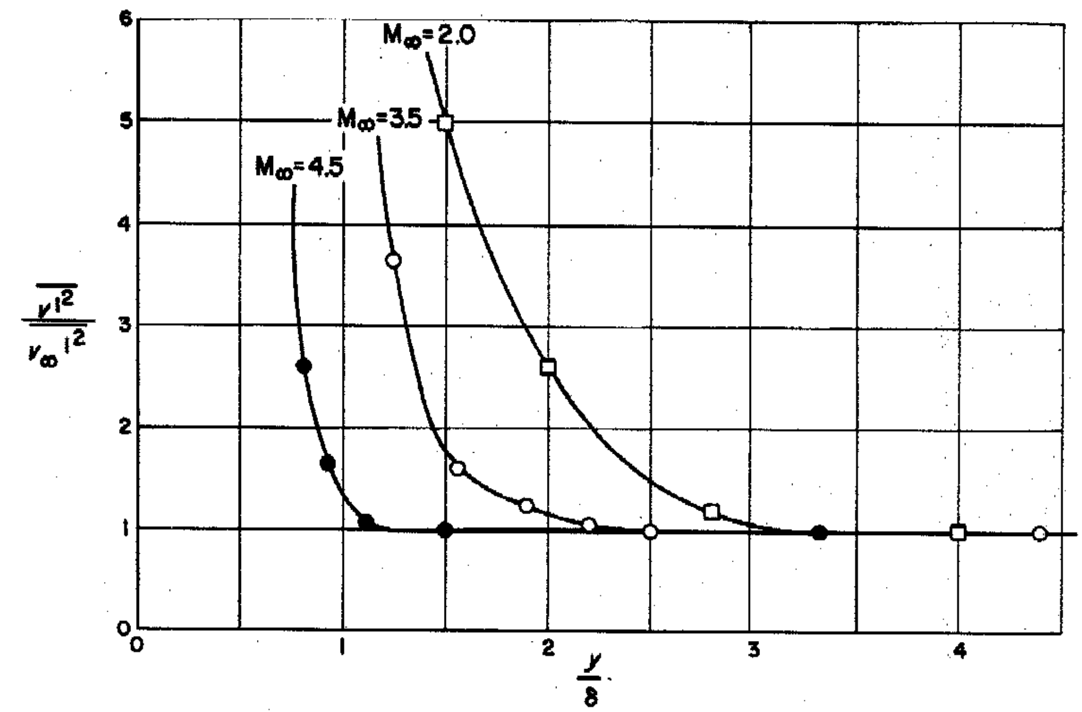

FIGURE 3
Fluctuation near the edge of the boundary layer.

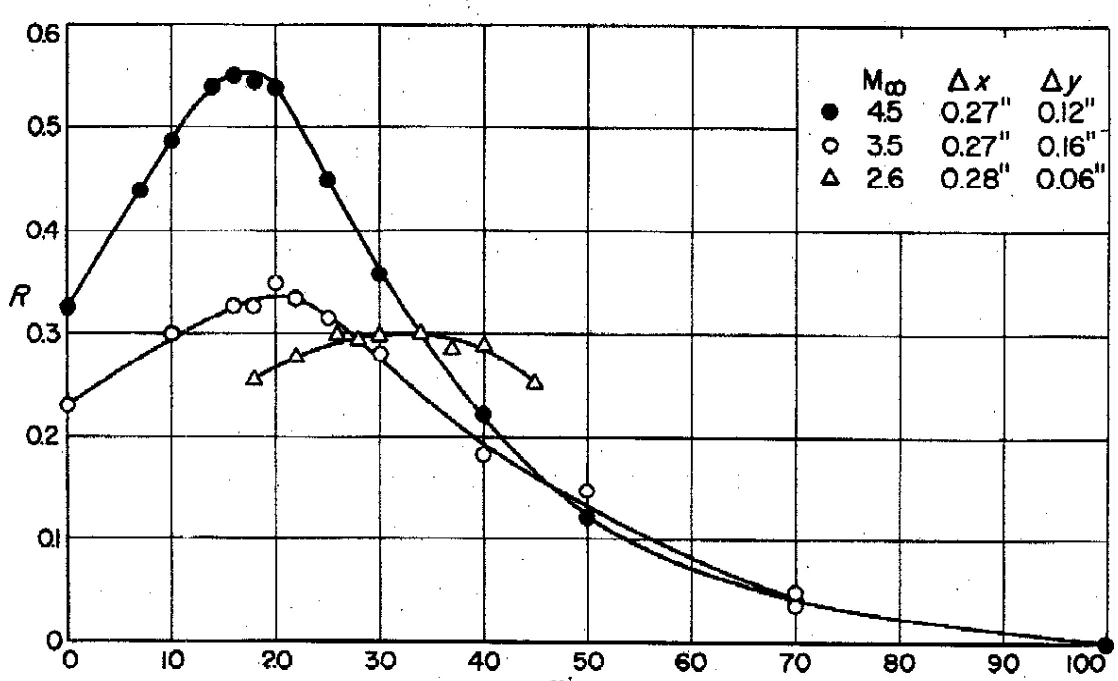

FIGURE 4
Space-time-correlation in far-field.

The directional characterictics of the field may also be investigated by measuring the space-time correlation of the pressure fluctuations. Figure 4 shows the result of such measurements for three Mach numbers. The two hot wires were behind each other at a distance ΔX apart, indicated in the figure; they were also displaced in the plane perpendicular to the flow direction sufficiently that no mutual interference was observed. It is seen that there exists a particular time delay τ_m for which the correlation is a maximum; or expressing it another way : there exists a preferred velocity $U_c = \frac{\Delta X}{\tau_m}$ with which the fluctuation patterns are convected downstream. Since the measurements are made several wavelengths away from the layer, one can assume the sound waves to be plane. Then the above result implies that the wave fronts have a preferred direction; as a matter of fact, with increasing Mach number more and more of the sound energy is oriented in one particular direction (the maxima of the correlation curves become stronger).

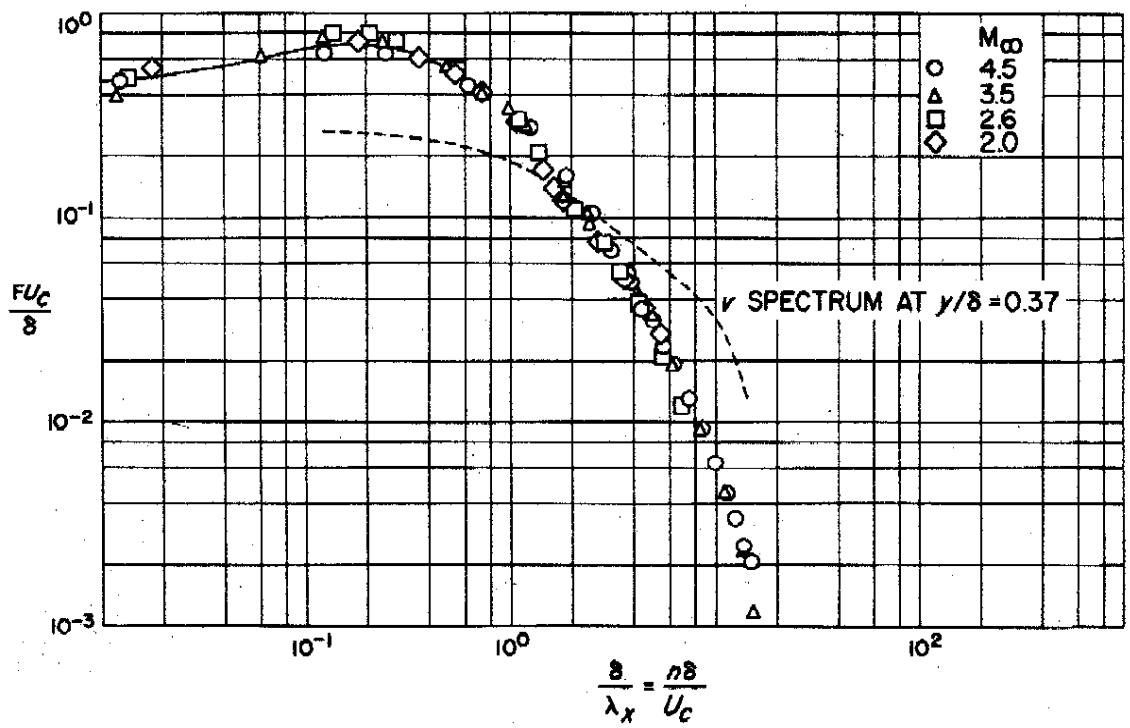

FIGURE 5
Energy spectrum of pressure fluctuations.

A consistent result is obtained if the spectra of the pressure fluctuation obtained at various Mach numbers are compared (the $\frac{Re}{in}$ of the tunnel, or approximately $U_\infty \frac{\lambda}{\nu}$ was held constant). Figure 5 shows that by choosing for the convection velocity the values obtained by the correlation method, the spectra exhibit a similarity throughout the number range. This implies that the directional characteristics of the sound waves of all wavelengths are the same. The similarity also implies that for a given boundary layer, the wavelengths $\lambda = \lambda_x \cos\theta$ decrease with increasing Mach number.

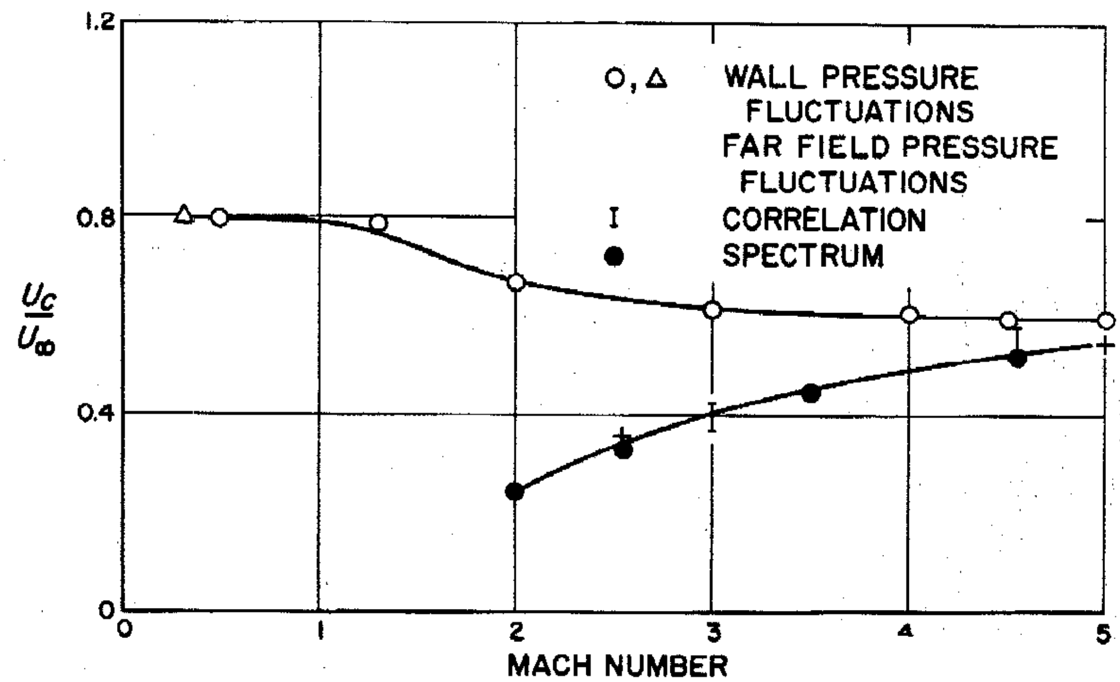

FIGURE 6
Convection velocity ratio for pressure fluctuations.

Figure 6 shows the variation of the convected velocities with Mach number. The upper curve corresponds to those obtained by observing the pressure fluctuations at the wall. As pointed out earlier, these fluctuations are believed to be produced mainly by the vorticity field within the layer, by the large-scale energy-carrying eddies. It is seen that for low speeds the convection velocity is $0.8\,U_\infty$, indeed the same as found by Favre for the large-scale eddies from the space-time velocity correlation measurements (Ref. 9). The lower curve shows the convected velocities of the sound field in the free stream. If one identifies these with some average velocities of sources producing the sound, one may explain the Mach number variation of these convection speeds in terms of Phillips' picture. According to Phillips, within the boundary layer only an inner layer which flows supersonically with respect to the free stream is the effective sound producer. The sound sources are then convected dowstream with some average velocity of this layer. Clearly, as the free stream Mach number increases, the relatively supersonic layer thickens, containing higher velocity sources.

From the above discussion of the theoretical and experimental results, one arrives at the following conclusion : Phillips' basic idea — namely, that the sound generation mechanism consists of a moving, spacially random, virtual wavy wall formed by an eddy pattern that is convected supersonically with respect to the free stream — is consistent with the main features of the sound field found experimentally. Such a virtual wall radiates a sound far-field that is homogeneous and has certain directional properties described earlier. However, it seems that the experimental Mach numbers are not high enough to yield the same functional behavior of the sound intensity with Mach number as predicted by the asymptotic theory. The subsonic region (as shown in Fig. 2) even for

M = 5 extends over half of the boundary layer. This implies that the sound fluctuations produced by the virtual wall will be attenuated in the subsonic region adjacent to this wall as they are radiated out in the far-field, the attenuation being much higher for the large wave numbers. This is believed to be partly the reason that the sound spectrum in the far-field contains much less energy in the high-frequency region than the spectrum of the generating function v' (see dashed line in Fig. 5).

If one now adopts the over-simplified point of view that all the sound is produced in a layer near the wall, the average velocity of which is U_c (the averaging is taken spacially across the layer), one may study the equivalent problem of the sound field produced by a randomly wavy wall moving with a relative Mach number

$$M_r = \frac{U_\infty - U_c}{a_\infty} = \frac{U_r}{a_\infty}$$

From the well-known potential solution, one obtains

$$\frac{\overline{p'^2}}{\gamma^2 \bar{p}^2} = \frac{\overline{v'^2}}{U_r^2} \frac{M_r^4}{M_r^2 - 1} \int \psi(\xi, \eta, \tau)\, d\xi\, d\eta\, d\tau$$

where $\psi(\xi, \eta, \tau)$ is a space-time correlation function. ($\xi = x_1 - x_2$, $\eta = z_1 - z_2$ in the plane of the layer; τ is the time delay $t_1 - t_2$.) Since $M_r = \left(1 - \frac{U_c}{U_\infty}\right) M_\infty = \frac{1}{\cos\theta}$, it is easy to show that the above expression becomes

$$\frac{\overline{p'^2}}{\gamma^2 \bar{p}^2} = \frac{\overline{v'^2}}{U_\infty^2} \frac{M_\infty^2}{\sin^2\theta} \int \psi(\xi, \eta, \tau)\, d\theta\, d\eta\, d\tau$$

In order to estimate the space-time correlation, one has to make some assumptions on the statistical behavior of the wall wavyness. It is reasonable that the fluctuations are correlated only over a certain area, say $L_x L_z$, where L_x and L_z are integral lengths that scale with the boundary layer thickness. Furthermore, the correlation must depend on a time scale corresponding to the average life time of a " bump "; we assume for lack of better information that it scales with $\frac{\delta}{U_c}$. Thus we may write for the space-time correlation

$$\psi(\xi, \eta, \tau) = \frac{L_x L_z}{\delta^2} \frac{\tau U_c}{\delta} \delta_D(\xi)\, \delta_D(\eta)$$

where $\delta_D(\xi)$, $\delta_D(\eta)$ are Dirac delta functions. Finally, since $\overline{v'^2}$ scales with the friction velocity $U_\tau^2 = \frac{C_f}{2 U_\infty^2}$, we may write

$$\frac{\overline{p'^2}}{\gamma^2 \bar{p}^2} = \frac{\overline{v'^2}}{U_\infty^2} \frac{C_f}{2} \frac{L_x L_z}{\delta^2} \frac{\tau U_\infty}{\delta} \frac{U_c}{U_\infty} \frac{M_\infty^2}{\sin^2\theta}$$

This relation indicates that the pressure intensity varies with the square of the Mach number. In Fig. 7 a comparison is made between the measured rms pressure fluctuations $\tilde{p}$ in the far-field and this relation in which we assumed

$$\frac{\overline{v'^2}}{U_\tau} \sim 1, \quad \frac{L_x L_z}{\delta^2} \sim 10^{-2}, \quad \frac{\tau U_\infty}{\delta} \sim 1$$

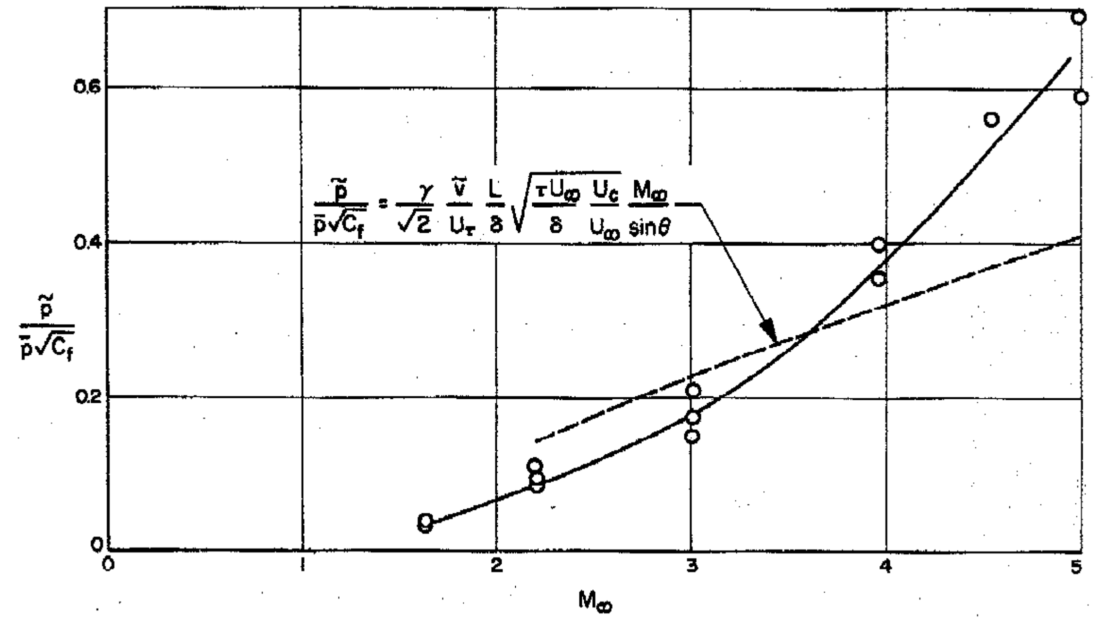

FIGURE 7
Far-field pressure fluctuations.

It is seen that the variation of the pressure fluctuations is much stronger than indicated by the above relation. The explanation may be due to the quantity $\frac{\tau U_\infty}{\delta}$ which, according to Ref. 9, varies rapidly across the boundary layer. In the Mach number range 1.6 to 5, the thickness of the radiating layer increases rapidly, and the increase in $\tilde{p}$ might be partially due to changes in $\frac{\tau U_\infty}{\delta}$.

The only conclusion we can draw from the above discussion is that in the interesting region of low supersonic Mach numbers there is no theory yet that describes properly the very fast increase of radiated energy of a boundary layer as the flow Mach number is increased. Additional measurements, especially that of space-time correlations of the v' fluctuation near the wall, are necessary to further clarify the problem.

With reference to the question of turbulence damping mentioned in the Introduction, it is possible, on the basis of the measurements described, to make a rough estimate of the energy loss due to radiation. The sound energy density in a moving medium may be written (Ref. 10)

$$E = \frac{\overline{p'^2}}{\rho a^2} \frac{V_p}{a}$$

where the phase velocity

$$V_p = a + \frac{\vec{U}_\infty \cdot \vec{a}}{a} = a + U_\infty \cos\theta = (1 + M_\infty \cos\theta)$$

or

$$E = \frac{\overline{p'^2}}{\bar{p}^2} \frac{\rho a^2}{\gamma^2} (1 + M_\infty \cos\theta)$$

The sound energy flux per unit area from the boundary layer

$$\vec{N} = E(\vec{a} + \vec{U}_\infty)$$

$$|N| = \frac{\overline{p'^2}}{\bar{p}^2} \frac{\rho a^3}{\gamma^2} (1 + M_\infty \cos\theta) \sqrt{M^2 + 1 + 2M_\infty \cos\theta}$$

or, in a nondimensional form,

$$\frac{N}{\rho U_\infty^3} = \frac{\overline{p'^2}}{\gamma^2 \bar{p}^2} \frac{(1 + M_\infty \cos\theta) \sqrt{M^2 + 1 + 2M_\infty \cos\theta}}{M_\infty^3}$$

At $M = 5$ the experiments give (remembering that the hot wire senses the radiation coming from four walls of the wind tunnel) :

$$\frac{\overline{p'^2}}{\bar{p}^2} = 0{,}6 \times 10^{-4} \qquad \cos\theta = -0.45$$

and therefore

$$\frac{N}{\rho U_\infty^3} = 1{,}4 \times 10^{-6}$$

This energy flux may be compared to the total work done by the wall shearing stress $W = \tau_w U_\infty$. In a nondimensional form

$$\frac{W}{\rho U_\infty^3} = \frac{\tau_w}{\rho U_\infty^2} = \frac{C_f}{2}$$

For the particular example, this value is approximately 3.3×10^{-4}. It is seen that at $M = 5$, the energy lost due to radiation is merely of the order of one percent of the total work done by the wall shearing stress. Thus it is quite clear that in order to resolve the question of complete turbulence damping, the radiation intensity variation with higher Mach numbers would have to be clarified.

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DISCUSSION DE LA COMMUNICATION DU Dr. LAUFER

Secrétaire scientifique : Dr. W. H. REID

Dr. P. G. SAFFMAN pointed out that there is a danger associated with the calculation of the sound radiated from a turbulent boundary layer. The difficulty arises from the result that the sound field from an infinite sheet of random sources is infinite and indeterminate if the source strengths are stationary random functions of time. If the sound from a distribution of random sources on a finite sheet of dimension L is evaluated, the mean-square fluctuations at a fixed distance from the sheet becomes infinite like log L as $L \rightarrow \infty$. The implication of this result is that PHILLIPS' analysis needs to be interpreted with care, since the coefficients of some of the higher order terms in the asymptotic expansion are infinite. However, this objection is not thought to be a serious objection to PHILLIPS' results, since the difficulty arises from the particular way in which the problem is formulated and solved mathematically, and can probably be circumvented. This point is being examined by Dr. Ff. WILLIAMS of the National Physical Laboratory, England. Dr. WILLIAMS has also considered the production of sound in supersonic flows, with an analysis much closer to the original formulation of LIGHTHILL than that of PHILLIPS; the results are similar, but there are differences in detail.

TURBULENCE IN CONDUCTING FLUIDS

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SOMMAIRE

Maintenant que l'extrême complexité de la théorie de la turbulence dans les fluides ordinaires a été révélée, il peut apparaître à beaucoup une extravagance téméraire d'aborder l'examen des fluides conducteurs de l'électricité. La situation est déjà assez mauvaise, pourquoi la rendre encore pire en autorisant les électrons, aussi bien que les molécules, à se mouvoir sans entraves ? A première vue, étendre toute théorie bien connue des fluides ordinaires à quelques-uns possédant cette dernière propriété semble refléter une autre explosion de cette panique contagieuse. Sur quelques points, l'accusation est justifiée. Une tendance s'est révélée de présenter des extensions directes de quelques-unes parmi les plus connues des théories mathématiques de la turbulence, nécessairement lourdes de formalisme mathématique, soutenues par des hypothèses d'une validité discutable et impliquant une série de conclusions dont la signification n'est comprise que partiellement. Mais l'aspect physique du sujet n'est pas encore suffisamment clarifié pour justifier une approche exclusivement mathématique.

Il est important à cette étape d'essayer de définir les sortes de situations physiques susceptibles de se produire, et c'est en partie mon but dans cette conversation.

Il y a relativement peu de publications à ce sujet et aucun travail expérimental n'a pratiquement été réalisé. Néanmoins, il y a deux raisons encourageantes de poursuivre ce sujet à fond.

— D'abord, dans les recherches concernant l'astrophysique, et la physique des plasmas, la présence de la turbulence est souvent supposée lorsqu'on ne peut pas expliquer les observations par une théorie « bien carénée ».

Il est cependant trop facile d'user, ou plutôt d'abuser, du mot « turbulence », comme d'une baguette magique, pour faire disparaître ce qui ne peut être interprété autrement.

Il est important d'arriver à des conclusions précises, quant à savoir quels phénomènes, dans des fluides conducteurs, peuvent être vraiment attribués à la turbulence, et quels phénomènes ne le peuvent pas.

— La seconde raison est peut-être plus académique. L'action de la turbulence sur une grandeur scalaire, telle que la température, qui est à la fois transmise et diffusée dans le fluide, est maintenant bien connue. Pour compléter le tableau, il serait intéressant de bien comprendre l'action de la turbulence sur une grandeur vectorielle, qui est de même transmise et diffusée. Le champ rotationnel est un exemple, mais il est trop particulier, car intimement lié au champ des vitesses.

Le champ magnétique dans un fluide conducteur est le parfait exemple de sujet de travail. Les lignes de force d'un champ magnétique, dans un fluide de conductivité infinie, sont transportées avec le fluide. Dans les fluides de conductivité finie, elles se diffusent à un taux dépendant de la grandeur de cette conductivité.

La situation est compliquée du fait que le champ magnétique exerce une force sur le fluide; il n'est généralement pas passif dynamiquement; mais dans certaines circonstances il sera possible de négliger cette force, et de se concentrer sur l'effet combiné de la convection et de la diffusion, dans un fluide turbulent, de propriétés statistiques connues.

SUMMARY

Now that the extreme complexity of the theory of turbulence in ordinary fluids has been revealed, it may seem to many a rash extravagance to admit to consideration fluids which conduct electricity. The situation is bad enough already — why make it worse by allowing electrons as well as molecules to move unfettered ? At first sight it seems to reflect another outburst of that infectious stampede to extend every known theory of ordinary fluids to those few with this " latest " property. To some extent, the accusation is justified. A tendency has revealed itself to present direct extensions of some of the better-known mathematical theories of turbulence, necessarily heavy with mathematical formalism, bolstered with assumptions of debatable validity, and carrying a trail of conclusions of partially understood significance. But the physics of the subject is not yet sufficiently clarified to justify an all-out mathematical approach. It is important at this stage to attempt to define the types of physical situation that may arise, and this is partly my aim in this talk.

There are relatively few published papers on the subject and practically no experimental work has been done. Nevertheless two compelling reasons can be given for pursuing the subject to its limit. Firstly, in astrophysics and in plasma physics research, the presence of turbulence is often inferred when observations cannot be explained by a streamlined theory. However it is too facile to use, or rather abuse, the word " turbulence ", like a magic wand, to dispel what cannot otherwise be understood. It is important to arrive at some precise conclusions as to what phenomena in conducting fluids can truly be attributed to the presence of turbulence, and what cannot. The second reason is perhaps more academic. The action of turbulence on a scalar quantity, such as temperature, which is both convected and diffused in the fluid is now fairly well understood. To complete the picture it would be satisfying to understand fully the action of turbulence on a vector quantity which is likewise convected and diffused. The vorticity field is an example, but it is too special, being closely related to the velocity field. The magnetic field in a conducting fluid is the perfect working example. The lines of force of a magnetic field in a fluid of infinite conductivity are convected with the fluid. In fluids of finite conductivity, they diffuse at a rate determined by the magnitude of this conductivity. The situation is complicated by the fact that the magnetic field exerts a force on the fluid — it is not in general dynamically passive; but in certain circumstances it will be possible to neglect this force, and concentrate on the combined effect of convection and diffusion in a turbulent fluid with known statistical properties.

2. The turbulent dynamo

The standard equations of magnetohydrodynamics can be conveniently written in terms of the fluid velocity $\mathbf{u}(\mathbf{r}, t)$ and the Alfvèn velocity at each point $\mathbf{h}(\mathbf{r}, t)$, which is simply proportional to the magnetic field $\mathbf{H}(\mathbf{r}, t)$:

$$\mathbf{h} = \sqrt{\frac{\mu}{4\pi\rho}}\,\mathbf{H} \tag{1}$$

where μ and ρ are the constant magnetic permeability and density of the fluid. In this notation the kinetic energy and the magnetic energy per unit mass are $\frac{1}{2}u^2$ and $\frac{1}{2}h^2$ respectively. The total pressure $\chi(\mathbf{r}, t)$ is the sum of the fluid pressure $p(\mathbf{r}, t)$ and the magnetic pressure $\frac{1}{2}\rho h^2$,

$$\chi = p + \frac{1}{2}\rho h^2. \tag{2}$$

The equations for $\mathbf{u}(\mathbf{r}, t)$ and $\mathbf{h}(\mathbf{r}, t)$ are then

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla\chi + \mathbf{h}\cdot\nabla\mathbf{h} + \nu\nabla^2\mathbf{u} \tag{3}$$

$$\frac{\partial \mathbf{h}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{h} = \mathbf{h}\cdot\nabla\mathbf{u} + \lambda\nabla^2\mathbf{h} \tag{4}$$

together with $\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{h} = 0$. (5)

Two diffusive constants appear, the kinematic viscosity ν, and the magnetic diffusivity λ. When $\lambda = 0$, one can deduce from equations (4) and (5) the well-known result that the flux of magnetic field through any circuit moving with the fluid remains constant, or equivalently that the lines of force move with the fluid and the strength of the magnetic field at any point moving with the fluid is proportional to the length of an element of the line of force through that point. I shall, for simplicity, restrict attention to homogeneous turbulence, and shall use the spectrum tensors $\Phi_{ij}(\mathbf{k})$ and $\Gamma_{ij}(\mathbf{k})$ of velocity and magnetic fields to describe the energy distributions in any steady state. Let it be our first aim to describe the development and steady state form of these spectra, given certain gross conditions defining the various situations that may arise.

In 1950, two irreconcilable theories were proposed, the one by Batchelor [1], the other by Biermann and Schlüter [2], to predict the development of an initially weak random magnetic field in a fluid in turbulent notion. To be fair, it must be stated that no fully convincing argument has yet been given to prove or disprove either theory. The matter is of fundamental importance and it seems highly appropriate that the theories should be reviewed at this meeting at any rate to clarify the points at which they diverge, and perhaps to suggest some critical problem whose solution might finally distinguish between the two standpoints. Let me therefore recall the main points of these theories.

Batchelor exploited the analogy between equation (4) for the magnetic field and that for vorticity $\omega\,(= \nabla\wedge\mathbf{u})$ in a non-conducting fluid, viz.,

$$\frac{\partial \omega}{\partial t} + \mathbf{u}\cdot\nabla\omega = \omega.\nabla\mathbf{u} + \nu\nabla^2\omega \tag{6}$$

$$\nabla\cdot\omega = 0. \tag{7}$$

Vorticity is generated by the stretching of vortex lines as they are convected by the fluid motion and it is destroyed by viscous diffusion at high wave-numbers. These two processes are approximately in equilibrium. In the same way, magnetic energy is generated by the stretching of magnetic lines of force in so far as they are convected by the turbulent motion. It is to be expected therefore that those statistical properties of the magnetic field that depend only upon this stretching mechanism will in time approximate to the corresponding statistical properties of the vorticity field. If $\lambda = \nu$, the conductive diffusion of lines of force is then just rapid enough for the magnetic field spectrum (like the vorticity spectrum) to remain approximately steady. If $\lambda > \nu$, conduction wins over stretching and the field decays to zero, while if $\lambda < \nu$, conduction is less important and the field increases in intensity. When λ is only slightly less than ν, it is not clear whether an all-round decrease of scale together with increased Ohmic dissipation limits the growth of the field, or whether it is the Lorentz force which modifies the straining motion and so limits the growth. Buth when $\lambda \ll \nu$, Batchelor argued that conduction alone would be of small importance and that the magnetic energy level must increase until magnetic stresses are comparable with the dynamic stresses governing the smallest turbulent eddies in which most of the vorticity is concentrated, that is until the mean magnetic energy per unit mass $\frac{1}{2} h^2$ is comparable with the kinetic energy per unit mass of the small-scale motion, $(\varepsilon\nu)^{1/2}$ (ε being the usual rate of dissipation of energy per unit mass).

Now BIERMANN and SCHLÜTER did not explicitly discuss the criterion for growth, but in any case they were considering a fluid, the interstellar gas, to which the condition $\lambda \ll \nu$ certainly applied, and they agreed with BATCHELOR that the mean magnetic energy would increase in these circumstances. However, it was their opinion that magnetic field components of all wave-numbers would be intensified, not only those with wave number near the viscous cut-off $\left(\frac{\varepsilon}{\nu^3}\right)^{1/4}$ where the vorticity is concentrated, as suggested by BATCHELOR. Briefly, they argued as follows. Consider eddies of dimension l larger than $\left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$ but small compared with the dimension L of the energy-containing eddies. The time-scale of such eddies by Kolmogorovian analysis is $l^{2/3}\varepsilon^{-1/3}$. When $\lambda \ll \nu$ the magnetic lines of force are to a very good approximation carried by these eddies as well as by all the smaller eddies that are superimposed on them. One might therefore expect loops of magnetic field of dimension l to be, say, doubled in intensity in a time of order $l^{2/3}\varepsilon^{-1/3}$. Such intensification could then continue until equipartition of energy was established at this length-scale. Equipartition would in this way be established by degrees at smaller and smaller wave-numbers until finally the whole spectrum was thus partitioned. Investigation of this argument reveals that although BIERMANN and SCHLÜTER agreed that the condition $\lambda < \nu$ was sufficient for initial growth they would not admit its necessity. The criterion most appropriate to their type of argument was stated explicity by SYROVATSKY [3] in 1957, who argued that magnetic eddies of size l would grow provided the stretching term $\mathbf{h}\cdot\nabla\mathbf{u}$ of equation (4) was greater in order of magnitude than the conduction term $\lambda\nabla^2\mathbf{h}$.

i. e., if
$$\frac{u_l}{l} > \frac{\lambda}{l^2} \quad \text{or} \quad R_m(l) \equiv \frac{u_l\, l}{\lambda} > 1 \tag{8}$$
where $u_l\,(=(\varepsilon l)^{1/3})$ is the velocity in an eddy of size l and $R_m(l)$ is the magnetic Reynolds number for that length-scale. Substituting for u_l this condition gives
$$l > \left(\frac{\lambda^3}{\varepsilon}\right)^{1/4}, \tag{9}$$
a situation that can arise only if
$$L \gg \frac{\lambda^3}{\varepsilon}. \tag{10}$$
The test case on which the two theories really collide is therefore when
$$\frac{1}{L} \ll \left(\frac{\varepsilon}{\lambda^3}\right)^{1/4} \ll \left(\frac{\varepsilon}{\nu^3}\right)^{1/4}. \tag{11}$$
For, by the first of these inequalities, the BIERMANN and SCHLÜTER attack predicts magnetic intensification at all wave numbers in the range $\left[\frac{1}{L}, \left(\frac{\varepsilon}{\lambda^3}\right)^{1/4}\right]$, while by

the second (tantamount to $\lambda \gg \nu$) the Batchelor attack predicts that all random magnetic field fluctuations ultimately decay to zero. If I use the semi-empirical formula for ε,

$$\varepsilon = \frac{u'^3}{L} \tag{12}$$

where u' is the r. m. s. velocity, and define the Reynolds number R and magnetic Reynolds number R_m of the turbulence by

$$R = \frac{u'L}{\nu}, \qquad R_m = \frac{u'L}{\lambda} \tag{13}$$

then the case (11) is defined in more fundamental terms by the inequalities

$$1 \ll R_m^{3/4} \ll R^{3/4} \tag{14}$$

Let me digress for a moment in order to consider this problem afresh in the light of related work on the spectrum of a scalar solute which is convected and diffused in a turbulent fluid. It is well known and understood that if a variation of temperature, say, is initially present in such a fluid, the turbulence rapidly mixes the temperature distribution, increasing temperature gradients without limit until molecular conduction finally erases all trace of variation. If a steady distribution of heat sources is present on a large length scale, so that in effect a pulse of temperature variation is emitted in each small time interval, then there is established a steady spectrum of temperature variation whose form at large wave-numbers has been discussed and determined in divers circumstances by Obukhov [4], Corrsin [5], Batchelor [6], and Batchelor, Howells and Townsend [7]. Can the development of magnetic field variations be followed in the same way? Let us concentrate first on the most controversial case described by (11) or (14) and suppose again that magnetic variations are present at a length scale l. The ability of turbulence to mix the convected quantity (now a vector) is in no way reduced. The new feature is the intensification through stretching of the convected lines of force. In other words the field may initially increase, but the claim that its length scale at the same time on average decreases is no stronger than the same claim that is accepted for the scalar field. When the length scale of the magnetic field is reduced below $\left(\frac{\lambda^3}{\varepsilon}\right)^{1/4}$, conduction outweighs intensification, and converts all the magnetic energy into Joule heat, no matter how much intensification may have initially taken place. Thus the magnetic pulse disappears in this case like the scalar pulse although it initially grew in strength for the reason underlying Syrovatsky's argument. Of course here again if a large scale magnetic field is maintained (e. g. a constant magnetic field may be externally applied, or a random large-scale distribution of electromotive forces may be supposed present) then the turbulence will generate fluctuations whose intensity will be proportional to the applied field and whose spectrum should be easily obtainable. Knowing the spectrum, it is possible to calculate the increased dissipation and the eddy diffusivity of the turbulent fluid. Thus if the magnetic fluctuation spectrum increases as $k^{+1/3}$ like the vorticity spectrum in the range where neither viscosity nor conductivity is important, i. e. up to the wave number $\left(\frac{\varepsilon}{\lambda^3}\right)^{1/4}$, and falls off rapidly beyond this wave number, then it can be shown that the eddy diffusivity is approximately equal to the ordinary diffusivity (λ) multiplied by the 5/2 — power of the magnetic Reynolds number.

3. Fluctuations at low magnetic Reynolds number when a uniform field is applied

It is noteworthy that the fluctuations will not be small compared with the applied magnetic field when the magnetic Reynolds number is large compared with unity. Hence any perturbation method which assumes that the fluctuations are small compared with the applied field can be valid only when the magnetic Reynolds number is smaller, and preferably much smaller, than unity. The perturbation approach was used by LIEPMANN [8] in 1952 and by GOLITSYN [9] in 1960, and although the condition $R_m \ll 1$ was not stated explicitly in either paper, it is apparently only to this case that the theories can be applied. Liepmann supposed that at time $t = 0$ a constant field $\mathbf{h}_0$ is switched on in a fluid in turbulent motion, and he derived the time development of the spectrum of the field fluctuations $\mathbf{h}_1$, that are generated, on the assumption that these always remain small compared with $\mathbf{h}_0$:

$$|\mathbf{h}_1| \ll |\mathbf{h}_0| \tag{15}$$

If this is true, then the equation for $\mathbf{h}_1$ becomes approximately

$$\frac{\partial \mathbf{h}_1}{\partial t} = \mathbf{h}_0 \cdot \nabla \mathbf{u} + \lambda \nabla^2 \mathbf{h}_1 \tag{16}$$

In terms of the Fourier coefficients of the fields $\mathbf{u}$ and $\mathbf{h}_1$, defined by

$$u_i = \int p_i(\mathbf{k})\, e^{i\mathbf{k}.\mathbf{r}}\, \mathbf{dk} \tag{17}$$

$$h_{1i} = \int q_i(\mathbf{k})\, e^{i\mathbf{k}.\mathbf{r}}\, \mathbf{dk} \tag{18}$$

equation (16) may be written

$$\left(\frac{\partial}{\partial t} + \lambda k^2\right) q_i = (\mathbf{h}_0 \cdot \mathbf{k})\, p_i \tag{19}$$

with solution

$$q_i(\mathbf{k}, t) = (\mathbf{h}_0 \cdot \mathbf{k}) \int_0^t e^{-\lambda k^2 (t-\sigma)}\, p_i(\mathbf{k}, \sigma)\, d\sigma. \tag{20}$$

If we now suppose that the kinetic turbulence remains statistically steady so that Φ_{ij} is independent of t (thus requiring that the energy transferred to the magnetic field must remain small compared with the total energy of the turbulence) then the spectrum $\Gamma_{ij}(\mathbf{k}, t)$ of the field fluctuation can be explicitly derived :

$$\begin{aligned}
\Gamma_{ij}(\mathbf{k}, t) &= \overline{q_i(\mathbf{k}, t)\, q_j^*(\mathbf{k}, t)} \\
&= (\mathbf{h}_0 \cdot \mathbf{k})^2 \int_0^t \int_0^t e^{-\lambda k^2 (2t-\sigma-\sigma')}\, p_i(\mathbf{k}, \sigma)\, p_j(\mathbf{k}, \sigma')\, d\sigma\, d\sigma' \\
&= (\mathbf{h}_0 \cdot \mathbf{k})^2 \int_0^t \int_0^t e^{-\lambda k^2 (2t-\sigma-\sigma')}\, \Phi_{ij}(\mathbf{k}, \sigma - \sigma')\, d\sigma\, d\sigma' \\
&= (\mathbf{h}_0 \cdot \mathbf{k})^2 \int_0^t \int_0^t e^{-\lambda k^2 (\rho+\rho')}\, \Phi_{ij}(\mathbf{k}, \rho' - \rho)\, d\rho\, d\rho'
\end{aligned}$$

(writing $\rho = t - \sigma$, $\rho' = t - \sigma'$). (21)

The star indicates a complex conjugate, and the overbar an ensemble average. $\Phi_{ij}(\mathbf{k}, t)$ here represents the Fourier transform of the space-time velocity correlation and its time dependence is not in general known. Howver, in the case considered here ($R_m \ll 1$) the integral is dominated by values of ρ and ρ' smaller than any characteristic time of the turbulence, so that $\Phi_{ij}(\mathbf{k}, \rho'-\rho)$ may be replaced by $\Phi_{ij}(\mathbf{k})$, the velocity spectrum tensor, (the first term in an expansion in powers of $(\rho'-\rho)$).
Then

$$\Gamma_{ij}(\mathbf{k}, t) = (\mathbf{h}_0 \cdot \mathbf{k})^2 \frac{(1-e^{-\lambda k^2 t})^2}{\lambda^2 k^4} \Phi_{ij}(\mathbf{k}). \tag{22}$$

Golitsyn independently derived with an equivalent approximation the asymptotic form of this relationship assuming the Kolmogorov spectrum for isotropic turbulence,

$$\Phi_{ij}(\mathbf{k}) = C\,\varepsilon^{2/3} k^{-5/3} \cdot \frac{1}{4\pi k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \tag{23}$$

where C is a constant of order unity. Substitution in equation (22) with $t = \infty$, gives

$$\Gamma_{ij}(\mathbf{k}) = C\,h_0^2\,\varepsilon^{2/3} \cos^2\theta \cdot k^{-11/3} \cdot \frac{1}{4\pi k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \tag{24}$$

The factor $\cos^2\theta$, where θ is the angle between $\mathbf{h}_0$ and $\mathbf{k}$, represents the anisotropy that is to be expected because of the preferred direction along $\mathbf{h}_0$. The spectrum, averaged over any sphere in wave number space, falls off as $k^{-11/3}$, i. e., more rapidly than the $k^{-5/3}$ fall-off the velocity spectrum, because the conductive damping of fluctuations increases with wave number more rapidly than the intensification through stretching.

4. The uniform strain attack when $R_m \gg R$

The above analysis is, as already observed, valid only when $R_m \ll 1$. Let us now return to the other extreme case $R_m \gg R$ (i. e. $\lambda \ll \nu$), that is, the case in which an instability to small magnetic perturbation is to be expected, and examine the consequences of applying methods that have already succeeded when applied to scalar fields. In this case of high conductivity the length scale at which conduction becomes important must be small compared with the length scale at which viscous forces control the smallest turbulent eddies. Any element of volume of dimension small compared with $\left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$ is simultaneously convected, rotated and uniformly strained in the fluid motion, and it is plausible to suppose that the uniform strain is the chief agent in modifying any magnetic field distribution within the element. Batchelor (6) has determined the form of the scalar spectrum at wave-numbers large compared with $\left(\frac{\varepsilon}{\nu^3}\right)^{1/4}$ by considering the effect of a uniform straining motion,

$$\mathbf{u} = (\alpha x, \beta y, \gamma z), \qquad (\alpha + \beta + \gamma = 0) \tag{25}$$

on a random homogeneous distribution of the scalar. We may now ask, what is the effect of such a uniform strain on an initially weak random homogeneous magnetic field distribution ? The answer has been effectively given by Pearson [10] who showed that the action of uniform strain on a weak random homogeneous vorticity distribution was to increase the mean square vorticity without limit. The same mathematics applied to the present problem shows that uniform strain increases the magnetic energy without limit. This is consistent with the conclusion that when $\lambda \ll \nu$ the magnetic energy increases until the magnetic body force intervenes to restrict the growth. But at the same time it indicates that the assumption of uniform constant strain is perhaps inadequate to represent the effect of turbulence on the highest wave number components of the magnetic field. A thorough examination of the effect of Lorentz forces within such small volume elements undergoing uniform strain and also of the effect of allowing the rate of strain tensor to change slowly in time would throw much light on this problem.

5. **Turbulence driven by magnetic forces**

The essential problem of stationary magnetohydrodynamic turbulence is to follow the flow of energy in wave number space from the two sources (kinetic and magnetic) at low wave-numbers to the two sinks (viscous and conductive) at high wave-numbers. The relative strength of the two sinks is controlled by the ratio $\frac{\nu}{\lambda}$ which is therefore vital in determining the kinetic and magnetic spectra at large wave numbers. Similarly the relative strength of the two sources is equally critical in the specification of the problem. In those problems considered so far the kinetic source (K) has been supposed strong compared with magnetic source (M). Indeed even when $M = 0$ it is likely that a steady state with non-vanishing magnetic field can be maintained if $\lambda \ll \nu$. The other extreme case for which $K = 0$ and only a magnetic source is present is equally interesting, and indeed more relevant to plasma experiments in which strong applied magnetic fields are the only obvious source of energy for the turbulence that is inferred from photographs. This situation has been discussed for a geometry with cylindrical symmetry by Kovasznay [11] who considered extensions of Reynold's equation for mean quantities derivable from equations (3) and (4). The velocity fluctuations were estimated from the balance between Reynolds stress and magnetic stress terms from equation (3), and this led to an estimate of the induced mean electric field, $\overline{\mathbf{u} \wedge \mathbf{h}}$, due to motion across applied field lines. Knowing the mean current, the effective eddy conductivity of the plasma follows; the value obtained by Kovasznay compared favourably with experimental estimates.

The situation considered by Kovasznay ($K = 0$, $\lambda \gg \nu$) is in a sense complementary to that considered by Batchelor ($M = 0$, $\nu \gg \lambda$). Kovasznay's work was motivated by observations of spontaneous turbulence in the presence of applied fields; Batchelor's by the widespread astrophysical phenomenon of spontaneous magnetic fields in the presence of background turbulence. The kinetic and magnetic spectra for Batchelor's case and for the isotropic analogue of Kovasznay's case are sketched in figures 1 and 2, in which this complementarity is pronounced.

To make the foregoing picture of magnetohydrodynamic turbulence less impressionistic, some experimental results are very much required. For example the determination of the amplification factor of a weak applied field in the case $R \gg R_m \gg 1$ would be sufficient to distinguish between the Batchelor standpoint and that of Biermann and Schlüter. The condition $R_m \gg 1$ is unfortunately hard to realise in laboratory conditions, but it

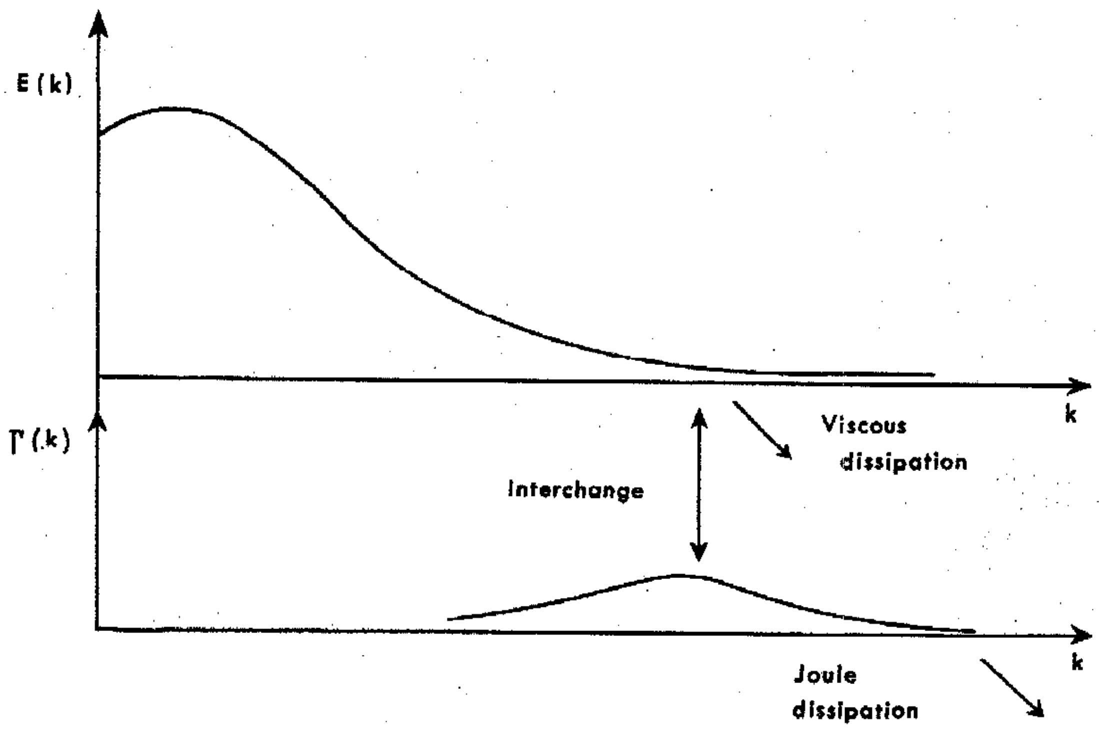

FIGURE 1

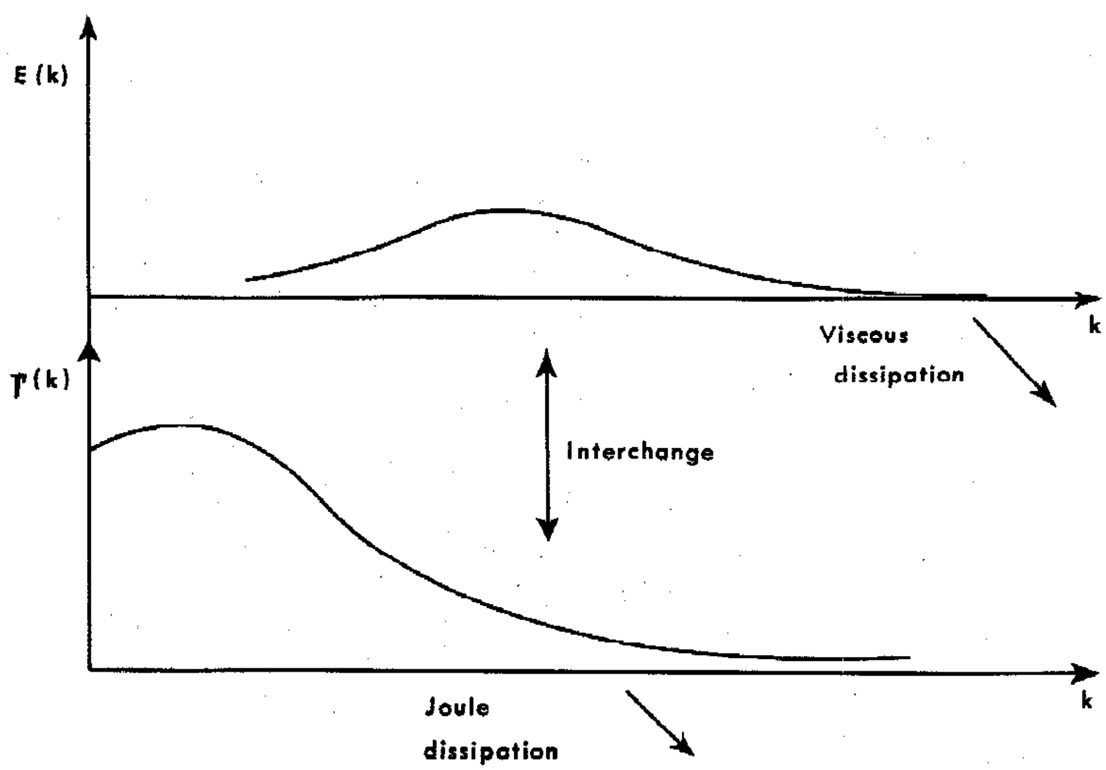

FIGURE 2

may not remain hopelessly beyond experimental technique. The less interesting situation $R_m \ll 1$ offers more scope for experiment. The way has been pioneered by MURGATROYD [12] who demonstrated that turbulence in channel flow of mercury could always be eliminated by applying a sufficiently strong transverse magnetic field. It would be valuable to determine the modification of the turbulence spectrum in the presence of an increasing field before the elimination is complete, and also to repeat the experiment with a longitudinal magnetic field (which does not directly distort the mean velocity profile) as well as with conducting fluids other than mercury. It is a little paradoxical that increasing the magnetic source of energy in Murgatroyd's experiment results in the suppression of turbulence. The reason is that in increasing the applied field a more effective vehicle is supplied for the immediate transfer of energy from the two sources (applied field and pressure drop in this case) to the conductive sink which drains energy efficiently at length scales of the order of the channel diameter. This reasoning only applies when $R_m \ll 1$. Kovasznay's contrasting picture of magnetic-driven turbulence is then relevant to the case $R_m \gg 1$, a condition that did indeed apply in the type of turbulent plasma that he considered.

Let me conclude by summarising the above observations in the following rough classification of types of stationary magnetohydrodynamic turbulence together with the chief situations in which each type may arise.

(a) Kinetic source dominant, weak applied field, $K \gg M$.

(i) $R_m \ll 1$: Small field fluctuations only generated by turbulence (ionosphere, turbulent mercury, liquid sodium etc.)

(ii) $1 \ll R_m \ll R$: Applied field intensified to level controlled by conduction (stellar interiors, regions of the ionosphere)

(iii) $R_m \gg R$: Equipartition at high wave-numbers, even if M is zero (HII regions of interstellar gas)

(b) Magnetic source dominant : strong applied fields, $M \gg K$

(i) $R_m \ll 1$: Suppression of turbulence (experiments on mercury in an increasing field)

(ii) $R_m \gg 1$: Magnetic driven turbulence (hot plasma, stellar interiors)

This is only a tentative scheme of limiting cases. A more thorough examination of the particular cases $K = M$ and $R_m = 1$ might also throw light on the general situation.

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COMMENTAIRE DE LA SECTION

TURBULENCE EN MILIEU COMPRESSIBLE ET ÉLECTRO-CONDUCTEUR

Prof. Leslie S. G. KOVASZNAY, Président

Une session spéciale était consacrée aux effets de la compressibilité et de la présence d'un milieu possédant une conductibilité électrique.

Il semble être un peu prétentieux de s'occuper de ces complications quand la turbulence simple d'un fluide incompressible et non-conducteur présente elle-même des difficultés presque insurmontables.

De nombreuses raisons conduisent à exécuter des recherches dans ces domaines quelque peu ésotériques. Le bruit produit par l'écoulement turbulent est un problème pratique en ce qui concerne les avions à réaction. La turbulence magnéto-hydrodynamique semble devenir un obstacle important au développement des réacteurs thermo-nucléaires contrôlés. Mais, même du point de vue de la recherche de base, cette question est intéressante parce que le cas du fluide incompressible et non-conducteur peut être mieux compris en tant que cas limite du fluide compressible et conducteur.

Monsieur MORKOVIN a discuté les résultats obtenus dans la couche limite turbulente supersonique.

Les mesures de turbulence faites à l'anémomètre à fil chaud dans la couche limite nous ont surpris. Même à un nombre de Mach de 1.75 à 2.00, nous avons constaté que le mécanisme interne de la turbulence diffère peu de celui de la couche limite incompressible. Bien entendu, il y a des fluctuations d'entropie, et même des fluctuations de pression (des ondes acoustiques), mais la véritable turbulence qui possède une divergence nulle, c'est-à-dire la partie incompressible du champ de vitesse, change très peu.

Une des questions essentielles est le comportement des tensions de Reynolds en milieu compressible, et un choix convenable des lignes de courant moyennes la ramène au cas incompressible.

Les spectres des fluctuations ressemblent aussi fortement à ceux des couches limites incompressibles.

Ces fluctuations peuvent être décomposées en trois modes : le mode rotationnel, le mode d'entropie et le mode acoustique. Deux de ces modes sont paraboliques, autrement dit, obéissent à des équations du type conduction de la chaleur. Par contre le mode acoustique est hyperbolique, et obéit à une équation de propagation d'ondes.

Dans un écoulement où la région turbulente est bornée, comme par exemple une couche limite turbulente, ou un jet, ou un sillage, les ondes acoustiques engendrées au sein de la portion turbulente se propagent et peuvent être observées dans l'écoulement extérieur non turbulent.

Monsieur LAUFER nous a présenté les résultats de mesures des fluctuations acoustiques obtenues à l'extérieur de la couche limite supersonique, et a fait aussi la critique des théories existantes sur la production de bruit par la couche limite supersonique. La théorie asymptotique de Phillips (valable à un nombre de Mach infini) se trouve approximativement confirmée. D'ailleurs l'énergie rayonnée est très faible par rapport à la dissipation visqueuse, même à un nombre de Mach très élevé, et par exemple à $M = 5$, elle est de l'ordre de 1 %, ce qui constitue un résultat surprenant.

Diverses considérations sur la turbulence magnéto-hydrodynamique ont été présentées par Monsieur MOFFATT et j'ai apporté personnellement quelques preuves expérimentales de l'existence de la turbulence dans un plasma. Quand le milieu possède une conductibilité électrique, les équations dynamiques (de Navier Stokes) comprennent un terme supplémentaire traduisant la force de Lorentz, qui est une fonction quadratique du champ magnétique. Par contre, l'équation qui gouverne le champ magnétique est linéaire.

Le problème essentiel de l'augmentation de l'énergie magnétique totale par l'agitation de la turbulence cinétique n'est pas résolu d'une façon définitive. D'autre part, un progrès considérable a été apporté dans le cas où le Nombre de Reynolds magnétique est très inférieur au Nombre de Reynolds cinétique. Dans ce cas particulier, le champ magnétique peut être traité par une méthode analogue à celle utilisée pour la diffusion turbulente, à cette différence près que le champ magnétique est une quantité vectorielle transportée d'une façon passive, tandis que la chaleur, ou la concentration d'une matière qui diffuse sont des quantités scalaires. La question expérimentale qui s'avère la plus importante est de trouver des moyens pour réaliser un écoulement turbulent de plasma qui soit simple et bien défini.