SECTION

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TURBULENT PIPE-FLOW

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SOMMAIRE

Vue d'ensemble, à la lumière des techniques de mesures utilisées, des distributions expérimentales des valeurs moyennes, en particulier de la vitesse moyenne. Le comportement au voisinage des parois (loi à la paroi) et au centre du tube seront discutés.

Les mesures de NIKURADSE des distributions de vitesse moyenne sont encore les plus complètes. De plus, celles de REICHARDT, LAUFER, DEISSLER et quelques autres seront retenues.

Toutes ces mesures, montrent dans la région turbulente au voisinage des parois, une concordance approximative avec la distribution logarithmique, avec les valeurs universelles des deux constantes intervenant dans cette loi. Mais, une analyse plus poussée dénonce une variation notable des valeurs de ces deux constantes. Elles semblent montrer une certaine dépendance du nombre de Reynolds.

Un résultat analogue peut être obtenu lorsque l'on considère la distribution dite « velocity - defect ». Dans la région centrale, cette distribution de vitesse ne présente pas une forme parabolique bien définie, sauf en ce qui concerne celles de REICHARDT et PAGE. C'est ainsi qu'il en résulte une « viscosité turbulente » non constante. Là encore semble jouer l'influence du nombre de Reynolds.

Ainsi la similitude basée sur le nombre de Reynolds, et la fixité de la constante de Von Kármán ne paraissent pas assurées.

Les idées concernant le comportement d'une « viscosité turbulente » dans la région intermédiaire, et dans la sous couche visqueuse sont encore spéculatives.

La théorie implique pour un fluide incompressible une variation au voisinage de la paroi proportionnelle au moins au cube de la distance. On ne possède pas de mesures concluantes.

Les résultats expérimentaux plutôt rares sur les quantités relatives à la turbulence sont en général insuffisants et insuffisamment certains, en particulier lorsqu'on se place tout près de la paroi, pour donner une opinion décisive.

Dans le cas d'une paroi rugueuse, les données expérimentales sur la répartition des vitesses moyennes peuvent être amenées à coïncider avec la loi logarithmique en choisissant une origine appropriée pour la coordonnée normale à la paroi. Mais, ceci présente encore une incertitude.

D'autre part, les constantes intervenant dans l'expression de la répartition des vitesses moyennes s'avèrent dépendantes encore du nombre de Reynolds, et de la nature des rugosités des parois.

SUMMARY

Survey in the light of applied measuring techniques of measured distributions of mean values, in particular of the mean-velocity. The behaviour in the wall region (the law of the wall) and in the core region will be discussed.

NIKURADSE's measurements of mean-velocity distributions are still the most complete. In addition those by REICHARDT, LAUFER, DEISSLER and a few others will be considered.

All these measurements show rough agreement in the turbulent wall region with the logarithmic distribution with universal values of the two constants occurring in it. But a closer analysis reveals a noticeable variation in the values of the two constants. They seem to show still a dependency on Reynolds number. A similar result may be obtained when considering the velocity-defect distribution. In the core region the velocity-defect distributions, with the exception of those by REICHARDT and PAGE, do not show a definite parabolic shape. Thus they result in a non-constant eddy « viscosity ». Here too an effect of Reynolds number seems present. So Reynolds number similarity and the constancy of Von KÁRMÁN's constant appear not to hold strictly.

Speculative are still the ideas about the behaviour of an eddy « viscosity » in the buffer region and in the viscous sublayer. Theory requires for an incompressible fluid a variation at the wall with distance to the third power at least. No reliable measurements are available.

The rather restricted data on turbulence quantities are in general insufficient and not reliable enough, in particular those close to the wall, to give a decisive argument.

In the case of a rough wall experimental data on the mean-velocity distribution may be made in agreement with the logarithmic distribution by choosing a suitable origin for the coordinate normal to the wall. But this still presents an uncertain point. Furthermore the constants occuring in the expressions for the mean-velocity distributions are found to be still functions of Reynolds number and of the nature of the wall roughness.

Introduction

The steady, fully developed turbulent flow through a straight pipe has been the subject of many investigations during the last four to five decades (STANTON [1], NIKURADSE [2, 3], REICHARDT [4], PAGE [5], LAUFER [6], DEISSLER [7], NUNNER [8], ABBRECHT [9], and others). The majority of these investigations deal with the mean-velocity distribution and with the flow resistance. Only a few of them also consider the turbulence structure, LAUFER's measurements being the most complete.

These investigations have led to the following concept concerning the steady turbulent flow through a straight pipe. Close to the wall the flow is entirely determined by the conditions at the wall (law of the wall). At sufficiently high REYNOLDS numbers there is no direct viscosity effect on the flow in the turbulent region of the pipe (REYNOLDS number similarity); so if the difference in mean-velocity with its maximum value at the centre is rendered dimensionless with the wall-friction velocity, it should be a function of the relative distance to the centre alone. This so-called velocity defect is independent of the REYNOLDS number. Since the turbulent region extends to within the wall region, there must be an overlapping turbulent region where the velocity defect law as well as the law of the wall hold. This results in the logarithmic mean-velocity distribution.

The experiments seem to confirm the above concepts. The measured mean-velocity in the overlapping part of the wall region appears to be described satisfactorily well by the logarithmic distribution. Though there is a notable scatter, while the values as suggested by the various investigators for the two constants occurring in the logarithmic distribution (one of which is Von KÁRMÁN's universal constant) show a less satisfactory variation.

A closer study of the experimental data now reveals that the scatter of data appears not to be entirely random, but that more or less systematic deviations seem to exist, suggesting that the above concepts might be not strictly correct. These concepts result only in an approximate description of the actual phenomena. So the law of the wall and the velocity-defect law might be only of approximate value, and the Von KÁRMÁN's constant might be not a constant and REYNOLDS number similarity might not exist or at the most at much higher REYNOLDS numbers than assumed hitherto.

In this paper we shall reconsider the most important experimental evidence in the light of the assumptions made and which have led to the current concepts mentioned. At the same time we shall discuss the concept of an eddy-viscosity for describing the mean-velocity distribution, and the difficulties met with when one has to account for the conditions at a wall of an arbitrary roughness.

Present situation

In steady fully-developed turbulent flow through a pipe the flow conditions and flow pattern are on the average homogeneous in flow direction. The equations describing the mean motion reduce to

$$\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x} = -\frac{1}{r}\frac{d}{dr}(r\,\overline{u_r u_x}) + \nu\frac{1}{r}\frac{d}{dr}\left(r\frac{d\overline{U}}{dr}\right) \tag{1}$$

$$\frac{1}{\rho}\frac{\partial \overline{P}}{\partial r} = -\frac{1}{r}\frac{d}{dr}(r\,\overline{u_r^2}) + \frac{\overline{u_\Phi^2}}{r} \tag{2}$$

These equations are independant of x. Hence $\overline{P}$ is a linear function of x and $\frac{\partial \overline{P}}{\partial x} = \frac{d\overline{P}_w}{dx}$, where $\overline{P}_w$ is the mean static-pressure at the wall.

Integration of the equation of motion for the axial direction yields

$$\sigma_{x_r} = -\rho\,\overline{u_r u_x} + \mu\frac{d\overline{U}}{dr} = \frac{r}{2}\frac{d\overline{P}_w}{dx} = \frac{D}{4}\left(1-\frac{2y}{D}\right)\frac{d\overline{P}_w}{dx} \tag{3}$$

or

$$\sigma_{x_r} = \sigma_w\frac{2r}{D} = \rho u^{*2}\frac{2r}{D} = \rho u^{*2}\left(1-\frac{2y}{D}\right) \tag{4}$$

Hence

$$-\frac{1}{r}\frac{d}{dr}(r\,\overline{u_r u_x}) + \nu\frac{1}{r}\frac{d}{dr}\left(r\frac{d\overline{U}}{dr}\right) = \frac{4\sigma_w}{\rho D} = 4\frac{u^{*2}}{D} \tag{5}$$

or

$$-\frac{1}{\xi'}\frac{d}{d\xi'}\left(\xi'\frac{\overline{u_r u_x}}{u^{*2}}\right) + \frac{\nu}{u^* r}\frac{1}{\xi'}\frac{d}{d\xi'}\left(\xi'\frac{d}{d\xi'}\frac{\overline{U}}{u^*}\right) = 2 \tag{6}$$

where $\xi' = \frac{2r}{D}$.

This equation, and general dimensional arguments suggest the following general function for the mean-velocity :

$$\frac{\overline{U}}{u^*} = \varphi\left(\frac{u^* r}{\nu}, \frac{2r}{D}, \frac{2k}{D}, \alpha, \beta, \ldots\right) \tag{7}$$

where k is a roughness parameter of the wall having the dimensions of a length (e. g. average roughness height) and α, β ... other dimensionless roughness parameters (as shapefactor, distribution factor, etc.)

Law of the wall. — This law says that in a small region close to the wall the flow is solely determined by the wall conditions, i. e. the wall shear stress and geometrical configuration. This implies that the flow is independent of D and of the flow conditions in the core region, and that the local shear stress does not differ much from the wall shear stress (constant-stress region). Hence according to the law of the wall the mean-velocity distribution has the following functional form :

$$\frac{\overline{U}}{u^*} = f\left(\frac{u^* y}{\nu}, \frac{y}{k}, \alpha, \beta, \ldots\right) \tag{8}$$

where y is the distance from the wall.

If the wall is smooth or hydraulically smooth, the velocity parameter u^* and the length parameter $\frac{\nu}{u^*}$ are sufficient to describe the flow (similarity) :

$$\frac{\overline{U}}{u^*} = f_1\left(\frac{u^* y}{\nu}\right) \tag{9}$$

If the wall is hydraulically rough u^* and k are the velocity and length parameter, whereas in addition the flow should also depend on the other parameters α, β, etc. :

$$\frac{\overline{U}}{u^*} = f_2\left(\frac{y}{k}, \alpha, \beta, \ldots\right) \tag{10}$$

The fact that the flow conditions should be independent of the conditions beyond the wall region must also be reflected in the turbulence energy balance. There should be no appreciable transfer of energy between this region and parts farther from the wall (Townsend [10]). For the wall region outside the viscous sublayer the turbulence-energy equation reads

$$\frac{\overline{u_r u_x}}{u^{*2}} \frac{d}{dy^+}\left(\frac{\overline{U}}{u^*}\right) + \frac{1}{u^{*3}} \frac{d}{dy^+} \overline{u_r\left(\frac{p}{\rho} + \frac{u_i u_i}{2}\right)} - \frac{\nu\varepsilon'}{u^{*4}} = 0 \tag{11}$$

where

$$y^+ = \frac{u^* y}{\nu} \quad \text{and} \quad \varepsilon' = \nu \overline{\frac{\partial u_j}{\partial x_i} \frac{\partial u_j}{\partial x_i}}$$

Thus the second convective term should be small compared with the local turbulence-production term and the local dissipation term, the production term being only determined by the wall shear-stress and the local mean-velocity gradient.

Velocity defect law. — From the general function (7) it follows that :

$$\frac{\overline{U}_{max}}{u^*} = \varphi\left(\frac{u^* D}{2\nu}, \frac{2k}{D}, \alpha, \beta, \ldots\right) \tag{12}$$

It is now postulated that in the turbulence region the velocity defect is a function of the relative distance only :

$$\frac{\overline{U}_{max} - \overline{U}}{u^*} = \varphi_1\left(\frac{2r}{D}\right) \tag{13}$$

so that it does not further depend on $\frac{u^* D}{2\nu}$, $\frac{2k}{D}$, α, β ... It implies the assumption of Reynolds number similarity.

This should also apply to the turbulence shear stress

$$\frac{\overline{u_r u_x}}{u^{*2}} = \psi\left(\frac{2r}{D}\right) \tag{14}$$

This latter may easily be true, since at sufficiently high Reynolds-number the viscosity does not contribute noticeably to the local shear-stress. Hence :

$$-\frac{\overline{u_r u_x}}{u^{*2}} \approx \frac{\sigma_{x_r}}{\rho u^{*2}} = \frac{\sigma_{x_r}}{\sigma_w} = \frac{2r}{D}$$

If the arguments leading to the law of the wall and the velocity defect law are accepted, as mentioned in the introduction, a logarithmic velocity distribution results for the overlapping turbulent part of the wall region :

$$\text{For a smooth wall : } \frac{\overline{U}}{u^*} = A \; ln \frac{u^* y}{\nu} + B \tag{15}$$

$$\text{For a rough wall : } \frac{\overline{U}}{u^*} = A \; ln \frac{y}{k} + B' \tag{16}$$

Here the constant A is the reciprocal of the Von Kármán universal constant κ.

If the logarithmic distributions would apply to the whole turbulence region, a velocity defect in accord with (13) would be obtained. However the actual maximum velocity at the centre appears to be greater than according to (15) or (16). Hence the velocity defect with respect to the actual maximum velocity and valid for the overlapping part of the wall region reads :

$$\frac{\overline{U}_{max} - \overline{U}}{u^*} = -A \; ln \frac{2y}{D} + B^* \tag{17}$$

where according to the Reynolds number similarity A as well as B* should be universal constants.

As mentioned in the introduction the measured velocity distributions in a region close to the wall follow satisfactorily well the logarithmic distributions (15) or (16), though the values for the constants A, B or B′ as suggested by the various investigators diverge. The most complete measurements are those by Nikuradse [2, 3]. By way of example his results are shown in Fig. 1, Fig. 2, Fig. 3 and Fig. 4. For the smooth wall condition Nikuradse suggested two values for the constants A and B, namely $A = 2.5$ and $B = 5.5$ if the best straight line in the semi-logarithmic plot is drawn through all the data points and $A = 2.4$ and $B = 5.84$ if only the region close to the wall $\left(\frac{u^* y}{\nu} < 3\,000\right)$ is considered. For the rough wall condition Nikuradse suggested $A = 2.5$ and $B' = 8.48$.

These results and also those from other investigators seem to confirm to a reasonable degree the concepts of the law of the wall and the velocity-defect law and Reynolds number similarity. Also Laufer's measurements on the turbulence-energy balance as shown in Fig. 5 and Fig. 6, seem to confirm the idea that the transfer of energy by convection, the second term in (11), is negligibly small compared with the local energy production and dissipation terms. Though the separate convective terms referring to pressure and kinetic energy respectively are not negligibly small.

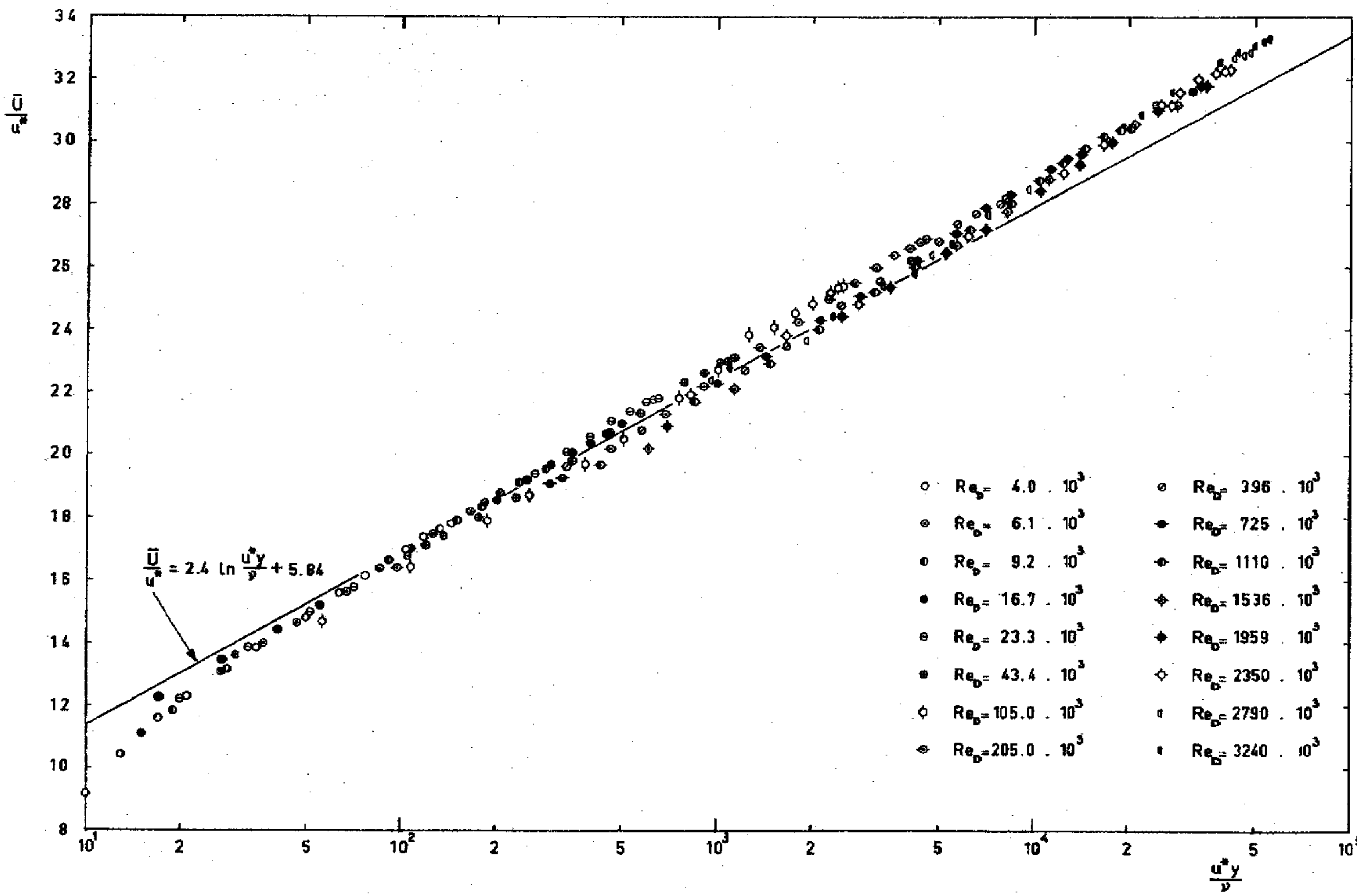

FIGURE 1
Nikuradse's measurements for a smooth pipe.

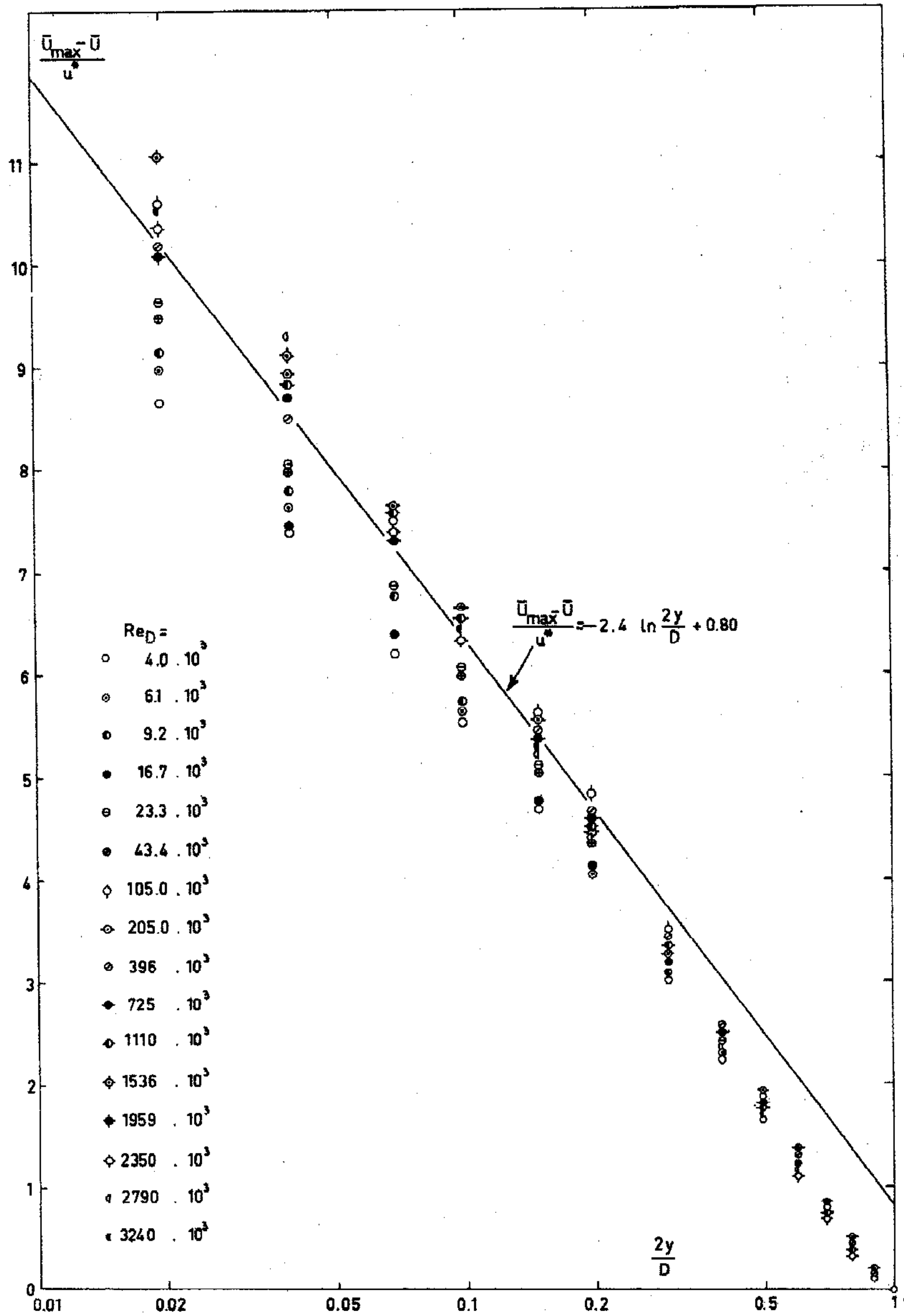

FIGURE 2
Velocity-defect according to Nikuradse's measurements for a smooth pipe.

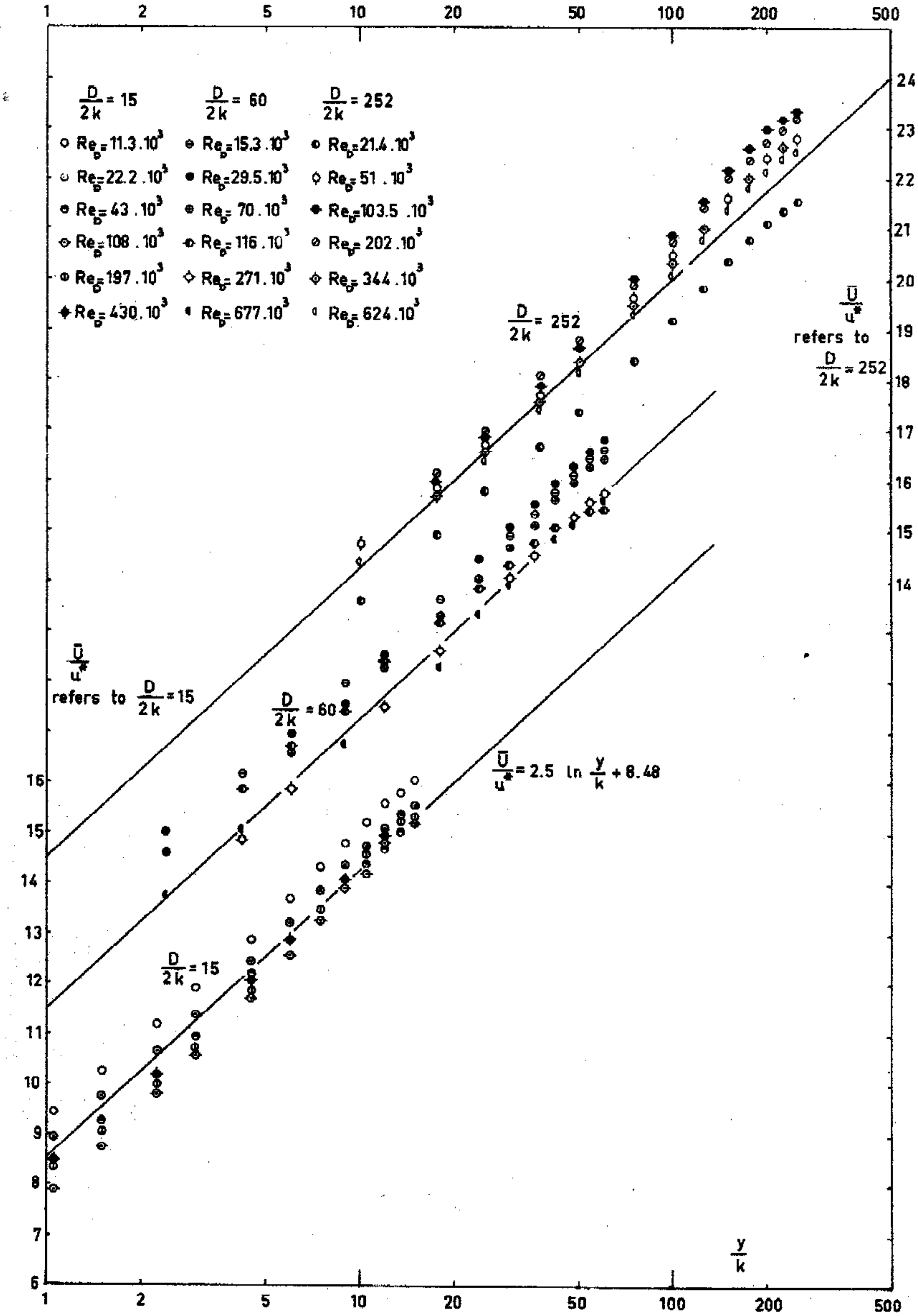

FIGURE 3
Nikuradse's measurements for a rough pipe.

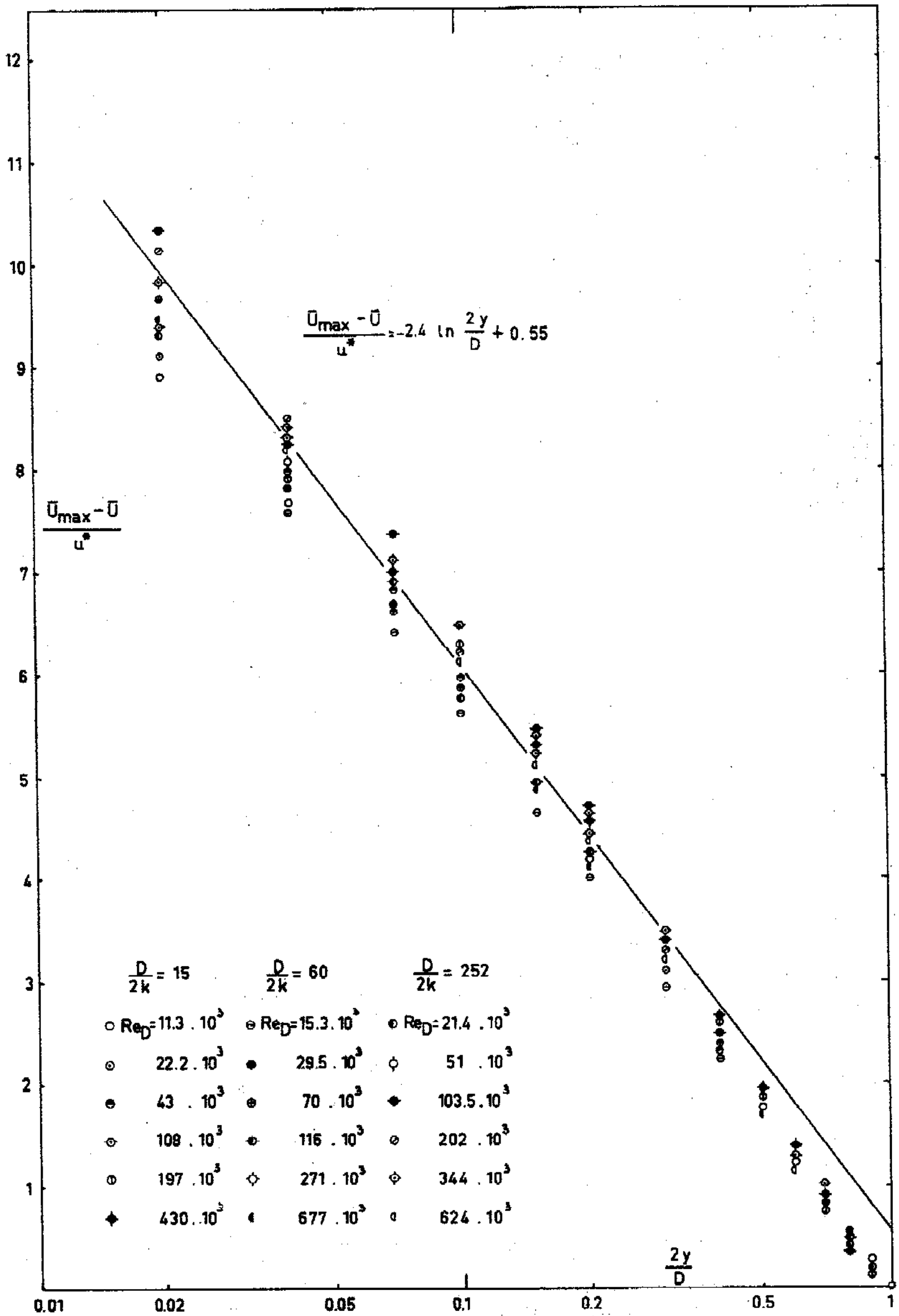

FIGURE 4
Velocity-defect according to Nikuradse's measurements for a rough pipe.

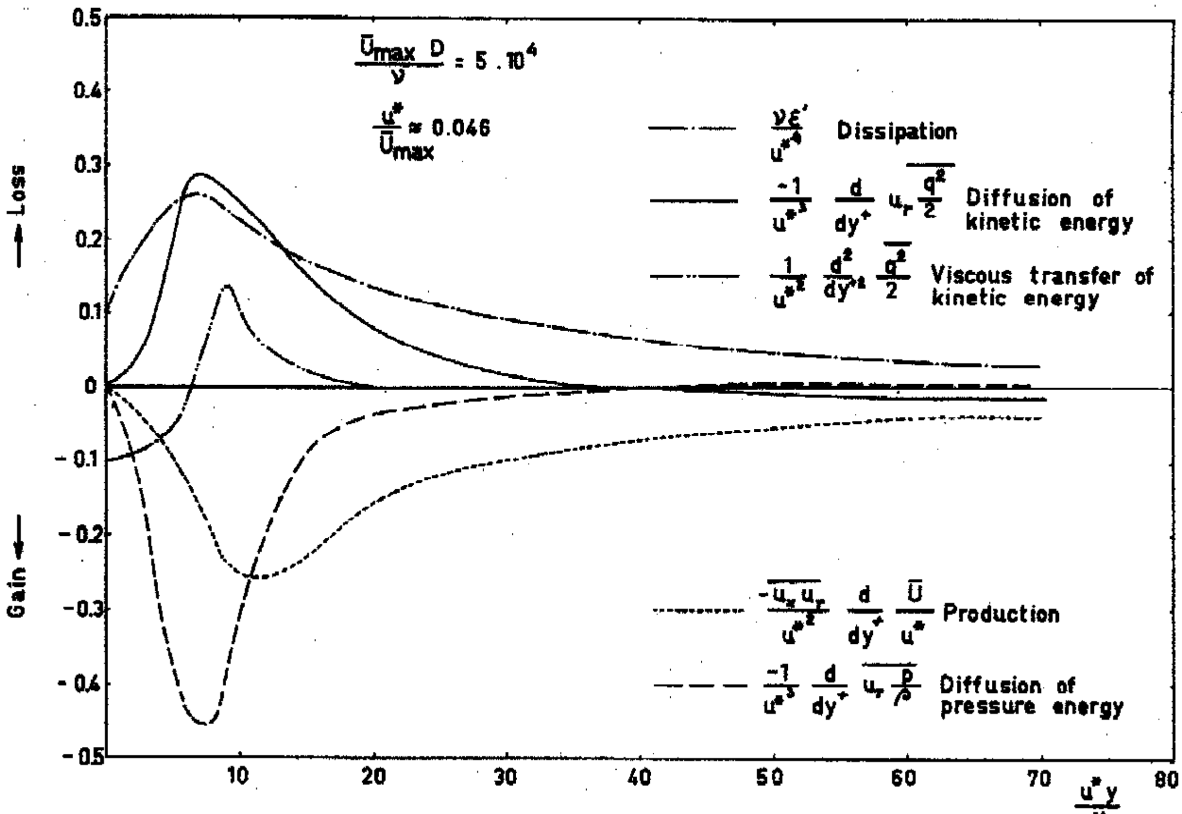

FIGURE 5
Energy balance in the wall region of pipe flow (LAUFER).

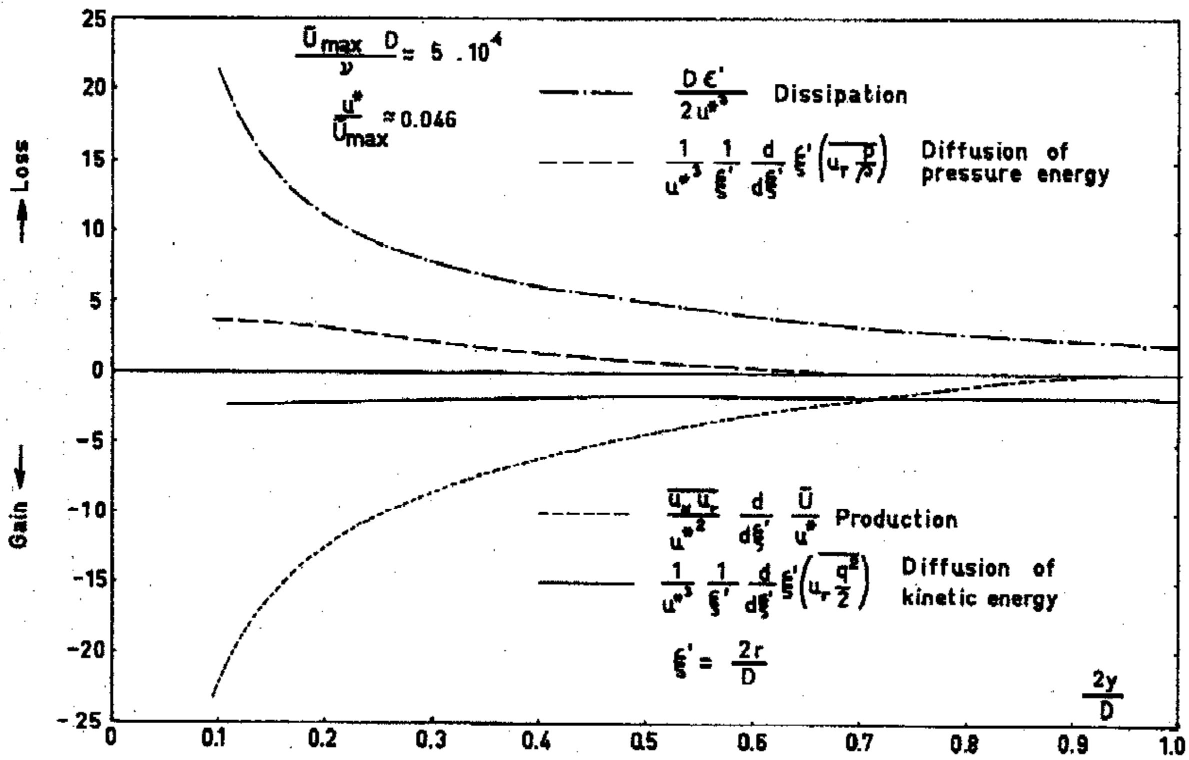

FIGURE 6
Energy balance in the core region of pipe flow (LAUFER).

The measurements on the mean-velocity distribution indicate that the logarithmic distribution applies in a wall region in which the shear stress varies by less than 10 to 20 percent of the wall shear-stress. TOWNSEND [10] has shown that at least 90 percent of the turbulence-energy production takes place in a region where the shear stress varies by less than 10 percent. If the convective term in the energy-balance equation is negligibly small the turbulence energy production and the dissipation should vary inversely proportional with the distance to the wall. This too seems to be confirmed by Laufer's measurements. Finally a 10 percent variation in the shear stress results only in a few percent effect on the mean-velocity distribution calculated on the assumption of a constant shear-stress, that is within the usual accuracy of the measurements.

Reconsideration of the mean-velocity distribution

Though the considerations given in the foregoing section seem to point towards the correctness of the concepts discussed and of the underlying assumptions, yet a closer examination of the available data throws some doubt on them. Consider for instance closely Nikuradse's results given in Figures 1 to 4. In Fig. 1 a systematic deviation from the mean general trend and depending on the pipe Reynolds number $Re_D = \overline{U}_{aver} \dfrac{D}{\nu}$ is noticeable. This means that the constants A and B occurring in the logarithmic distribution (15) are not independent of Re_D. The same can be observed with the velocity distributions measured by others at various values of Re_D. In the following we shall reconsider not only Nikuradse's data, but also those obtained by Reichardt, Page, Laufer, Deissler, Nunner and Abbrecht, all referring to smooth wall conditions. However, it may be worth while to give first a brief description of the experimental techniques applied in the most important experiments.

In his smooth-pipe experiments with water as the flowing fluid, Nikuradse used pipes of 1 cm, 2 cm, 3 cm, 5 cm and 10 cm I.D. The mean-velocity measurements were made in a section 0.1 to 0.2 mm downstreams of the pipe exit, the pipe having at least a length-diameter ratio of 70. Since Nikuradse did not observe a difference in static pressure between any point of a cross section measured with a static-pressure tube and a hole at the wall, he concluded the static-pressure distribution at the measuring section to be uniform. This calls up some doubt as to the accuracy of the static-pressure measurements, for due to the turbulence present in the flow through the pipe a variation of static-pressure should have been observed (see eq. (2)). Thus the mean-velocity distribution was measured with a total head tube (I.D. 0.21 mm and 0.30 mm) and a static hole of 0.8 mm diameter in the pipe flange at 2 mm radial distance from the jet boundary. Corrections have been made for the measurements near the wall for the effect of the finite hole diameter when the distance was smaller than half the internal total-head tube diameter. It turns out however that Nikuradse applied a constant shift of $\dfrac{u^* y}{\nu} = 7$ to make $U = 0$ at $y = 0$. This value corresponds with a correction required at $Re_D = 4.10^3$ and a total-head tube of 0.3 mm I.D. This procedure makes that the mean-velocity data very close to the wall have to be considered with some reserve.

Laufer experimented with an air-flow in a brass pipe of 24.7 cm I. D. and a length-diameter ratio of $\sim$ 50. The measurements were made in a cross-section 5 to 10 cm upstream from the exit at two Reynolds numbers ($\overline{U}_{max}\frac{D}{\nu}$ = 50,000 and 500,000). At the lower Reynolds number the mean-velocity was measured with a total-head tube and a separate static-pressure tube of 1 mm diameter placed 1 cm above the total-head tube. The total head tube had an I.D. of 0.85 mm, with the end flattened to a slit of 0.15 mm height. Near the wall corrections for the effect of turbulence have been applied

$$\left[\overline{U}_{corr} \simeq \overline{U}_{meas}\left(\frac{1-\overline{u^2}}{\overline{U}^2}\right)^{1/2}\right].$$

At the higher Reynolds number a hot-wire anemometer (Platinum-Rhodium 90/10, diameter 2.5 micron, sometimes 1.25 micron; wire length 0.25 to 0.625 mm for the turbulence measurements) was used for measuring the mean-velocity distribution near the wall. A correction for the effect of turbulence was applied by an approximate graphical method using the known static calibration curve and the known R.M.S. value of the voltage fluctuations. The maximum correction was roughly 10 percent, the true mean-velocity being higher than the observed. Measurements with the hot-wire anemometer near the wall at the lower Reynolds number resulted in much too low values after applying the correction. For the rest the hot-wire anemometer and total-head tube results were in good agreement. No corrections for the effect of the wall on the hot-wire anemometer measuments were made. Since the point measured nearest to the wall was about 0.05 mm an appreciable wall effect may be expected. From a comparison of the measured shear stress with the theoretical value it is concluded that deviations up to 5 to 10 percent are possible.

Deissler too experimented with air, but in a pipe of 2.2 cm I.D. and a length-diameter ratio of 100. The mean-velocity measurements were carried out with a total-head tube of 0.4 mm I.D. and tapered at the outside. For some runs a tube was used with an opening flattened down to 0.125 mm. No corrections are reported for turbulence effect and for finite hole diameter in case of measurements close to the wall. The minimum distance to the wall amounted to roughly 0.05 mm which is smaller than the flattened tube-hole dimension.

In his experiments with air Nunner used a pipe of 5 cm I.D. and a length-diameter ratio of 70. The mean-velocity measurements were done at the exit section of the tube with a total-head tube of 1 mm I.D. Again no corrections for the turbulence effect and for the finite hole-diameter effect of the total-head tube have been made, but the minimum distance of a measuring point to the wall was 1 mm.

Reichardt did his measurements in an air flow through a channel (25 $\times$ 100 cm) of 1600 cm length. Fine Pitot tubes as well as a hot-wire anemometer were used to measure the mean-velocity distribution. No dimensions of these instruments have been reported. The effect of the wall on the hot-wire anemometer was determined experidentally in a Poiseulle flow through a small channel (3 $\times$ 30 cm). The wall-friction velocity u^* was determined from pressure-drop measurements. At low velocities this procedure was not very accurate. Thus Reichardt corrected the $\frac{\overline{U}}{u^*}$ values by making them in agreement with Nikuradse's values at the larger values of $\frac{u^* y}{\nu}$, namely in the turbulent wall region.

Let us now consider Nikuradse's measurements with a smooth-wall tube. He made the measurements at 16 values of Re_D, the lowest value was 4.10^3 and the highest value 3240.10^3. In Fig. 7 the data as corrected by Nikuradse are shown for 10 out of the 16 values of Re_D. For the sake of clearness the data pertinent to a certain value of Re_D are shifted over a distance of 1 cm in vertical direction. The left-hand scale refers to the data at $Re_D = 4.10^3$, the right-hand scale refers to the data at $Re_D = 3240.10^3$. The straight line through the data at $Re_D = 23.3\ 10^3$ is Nikuradse's average line with $A = 2.4$ and $B = 5.84$.

In Fig. 8 the data for the velocity-defect as given by Nikuradse and pertinent to the same Reynolds numbers are given. These data have not been corrected for the effect of finite total-head tube diameter in the region close to the wall.

These two figures show that none of the measured mean-velocity distributions follow the average trend as given by Nikuradse, and obtained from all data at various Reynolds numbers together. Though there is some uncertainty in fixing the straight line through a series of points, because these points also have to deviate from the straight line towards the wall and outside the wall region, yet the individual straight lines show clearly a higher value for the slope than $A = 2.4$, and increasing with increasing Re_D. At the same time the value of the second constant B decreases ~~with increasing Re_D~~. At the lower values of Re_D the values of A obtained from the velocity-defect distributions differ from those obtained from Fig. 7, which might perhaps be attributed to the fact that the velocity-defect data were not corrected. The constant B^* also appears to decrease with increasing Re_D.

The same general trend, namely an increase of the constant A with increasing Re_D can also be observed from the data obtained by the other investigators, with the exception of Laufer's data where the value of A at the lower Reynolds number is higher than at the higher Reynolds number. Figures 9 and 10 show the results of Deissler's and Reichardt's measurements.

The general trend of variation of A, B and B^* with Reynolds number is shown in Fig. 11. If we compare the results of various investigators, there is quite a difference in values of A and B for the same Reynolds number. Figures 12 and 13 show the velocity-distribution and the velocity-defect distribution measured by various investigators and referring to roughly the same Reynolds number.

Complete series of velocity-distribution measurements at various Reynolds numbers and with varying roughness factor, have only been made by Nikuradse. As is well-known Nikuradse made the pipes artificially rough by glueing sand grains of a narrow sieve range on the inner pipe wall. For the mean-velocity measurements Nikuradse applied the same technique as in the case of his smooth-pipe measurements. He did not correct the measurements close to the wall for the effect of finite total-head tube diameter. So the data close to the wall are not very reliable. Another uncertainty is presented by the fact that the origin ($y = 0$) of the mean-velocity distribution is not well known. It should lie somewhere between the original smooth wall and the average crests of the roughness elements. Nikuradse determined the $y = 0$ position by filling the roughened pipe with water and by calculating the equivalent radius of a pipe having the same volume per unit length as the volume of water contained in the rough pipe per unit of length.

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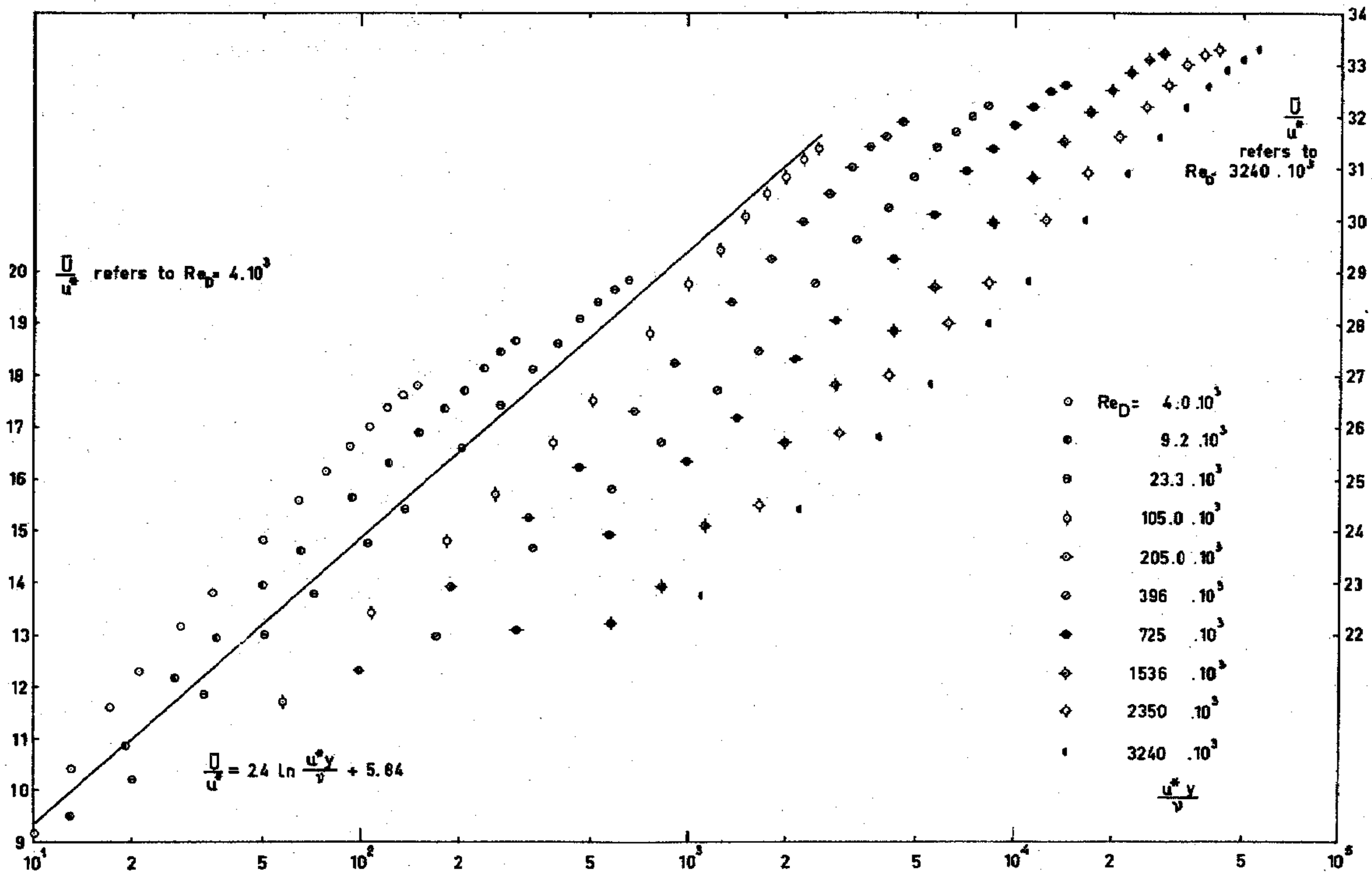

FIGURE 7
Mean-velocity distribution for a smooth pipe according to Nikuradse.

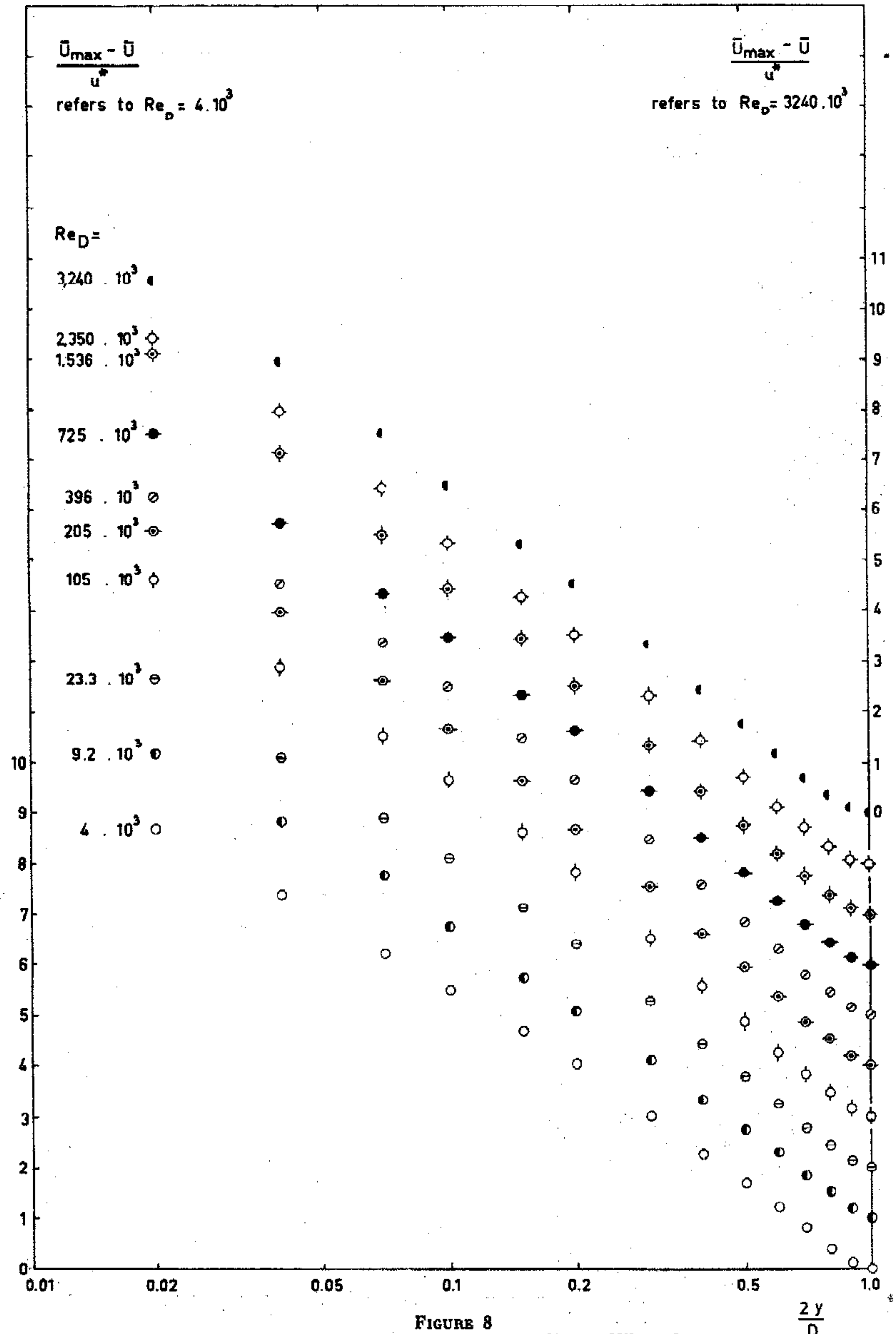

Figure 8
Velocity-defect for a smooth pipe according to Nikuradse.

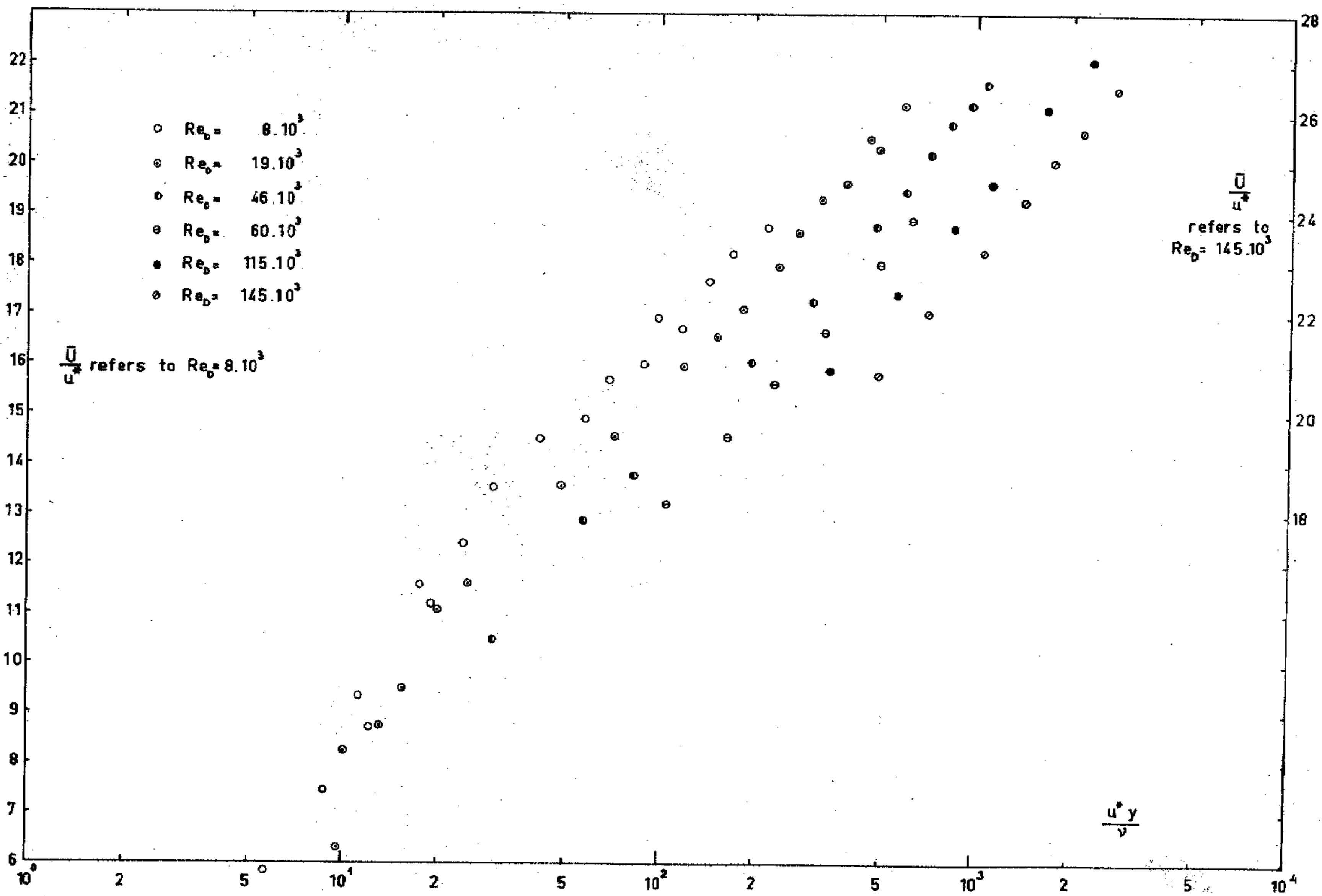

FIGURE 9
Mean-velocity distribution according to Deissler.

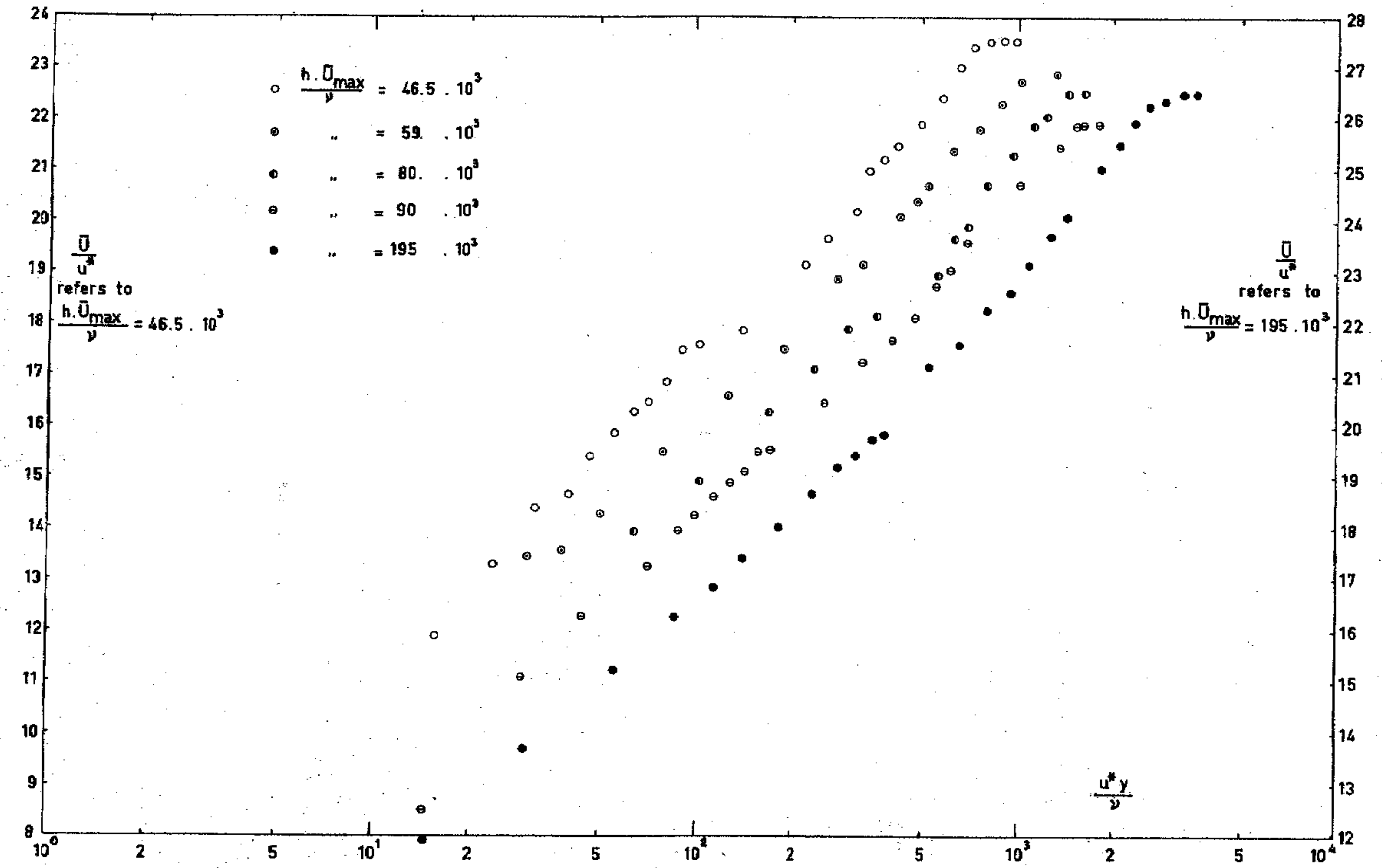

FIGURE 10
Mean-velocity distribution according to Reichardt.

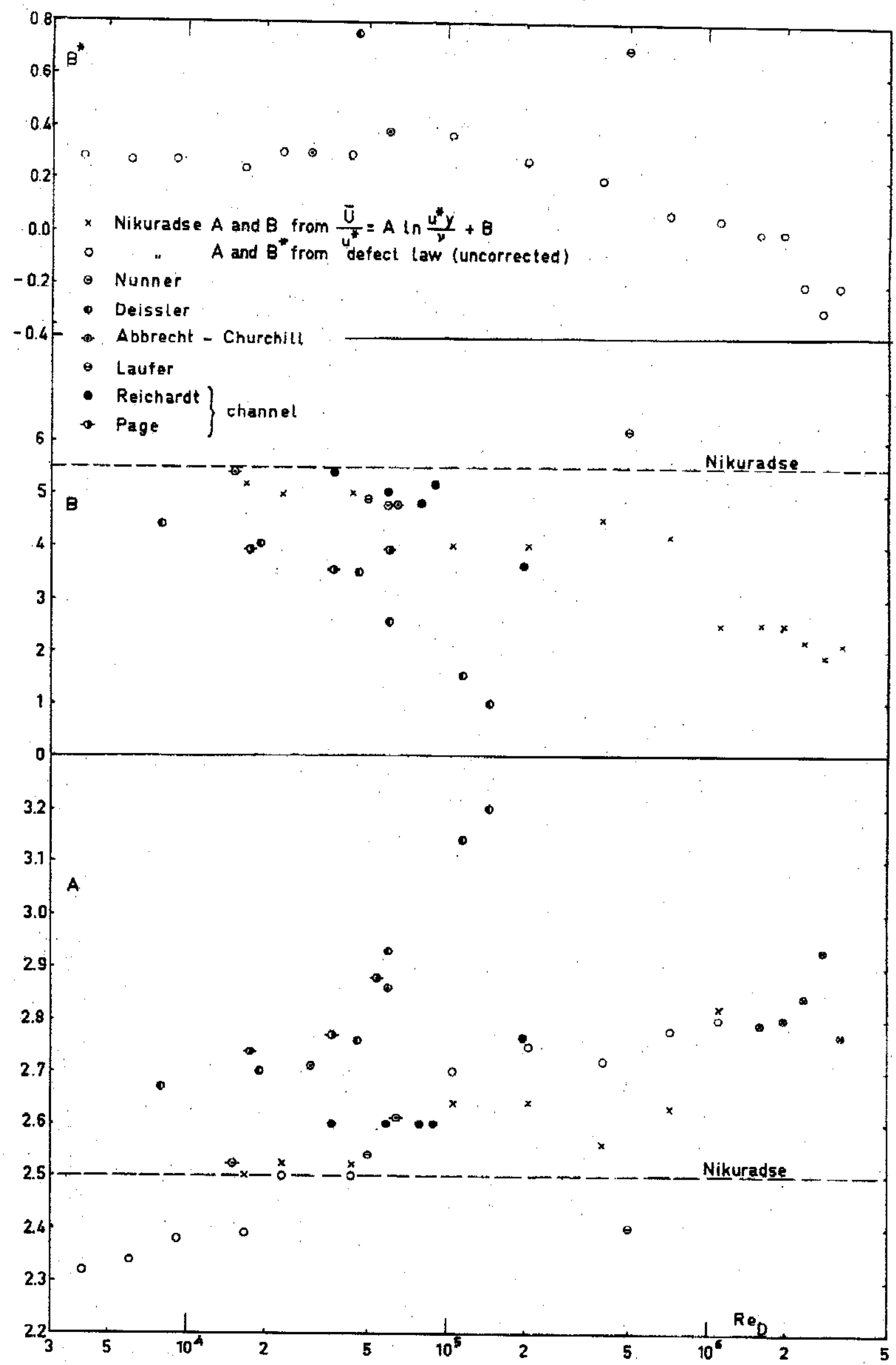

FIGURE 11
Variations of the constants A, B and B* with Reynolds number.

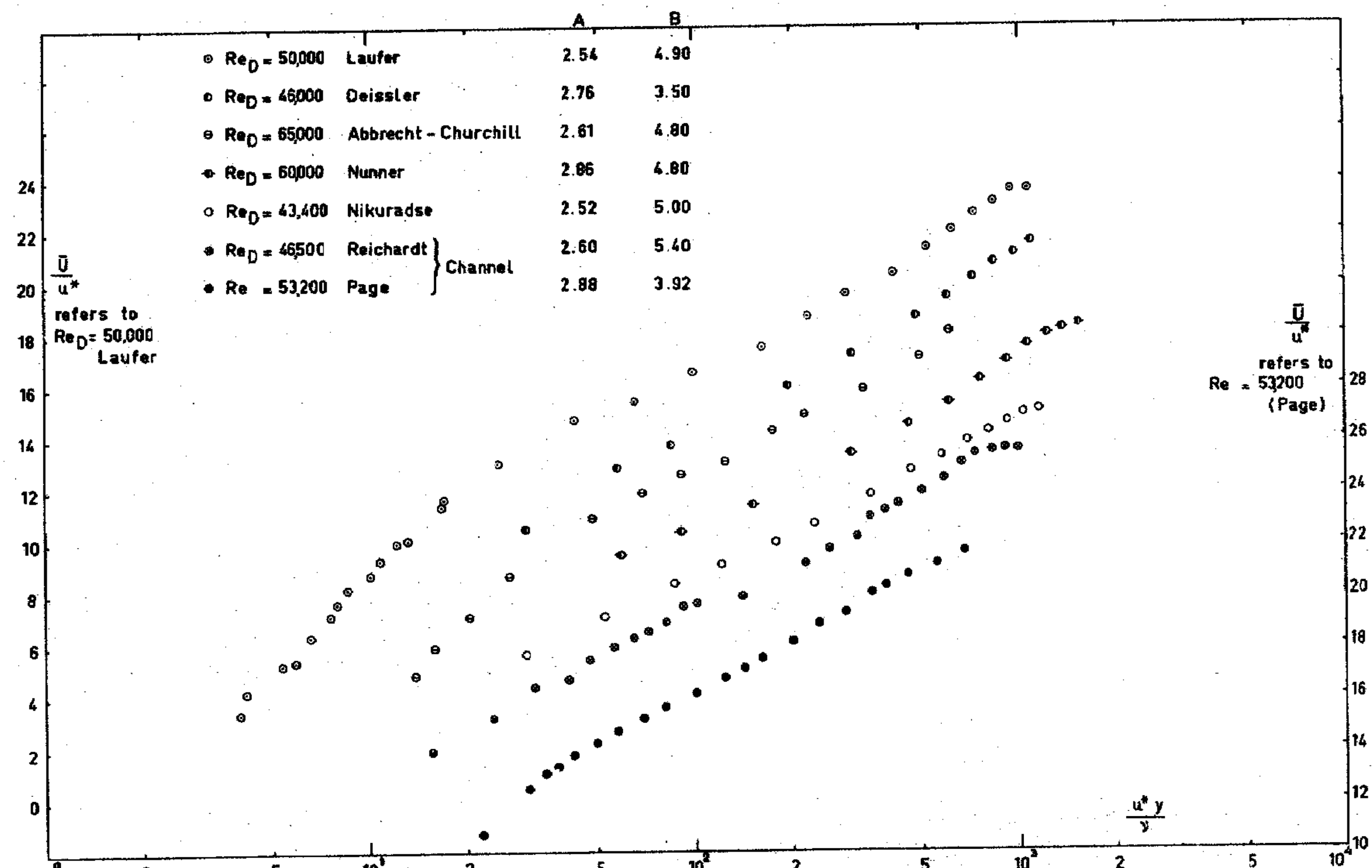

FIGURE 12.
Mean-velocity distribution measured by various investigators.

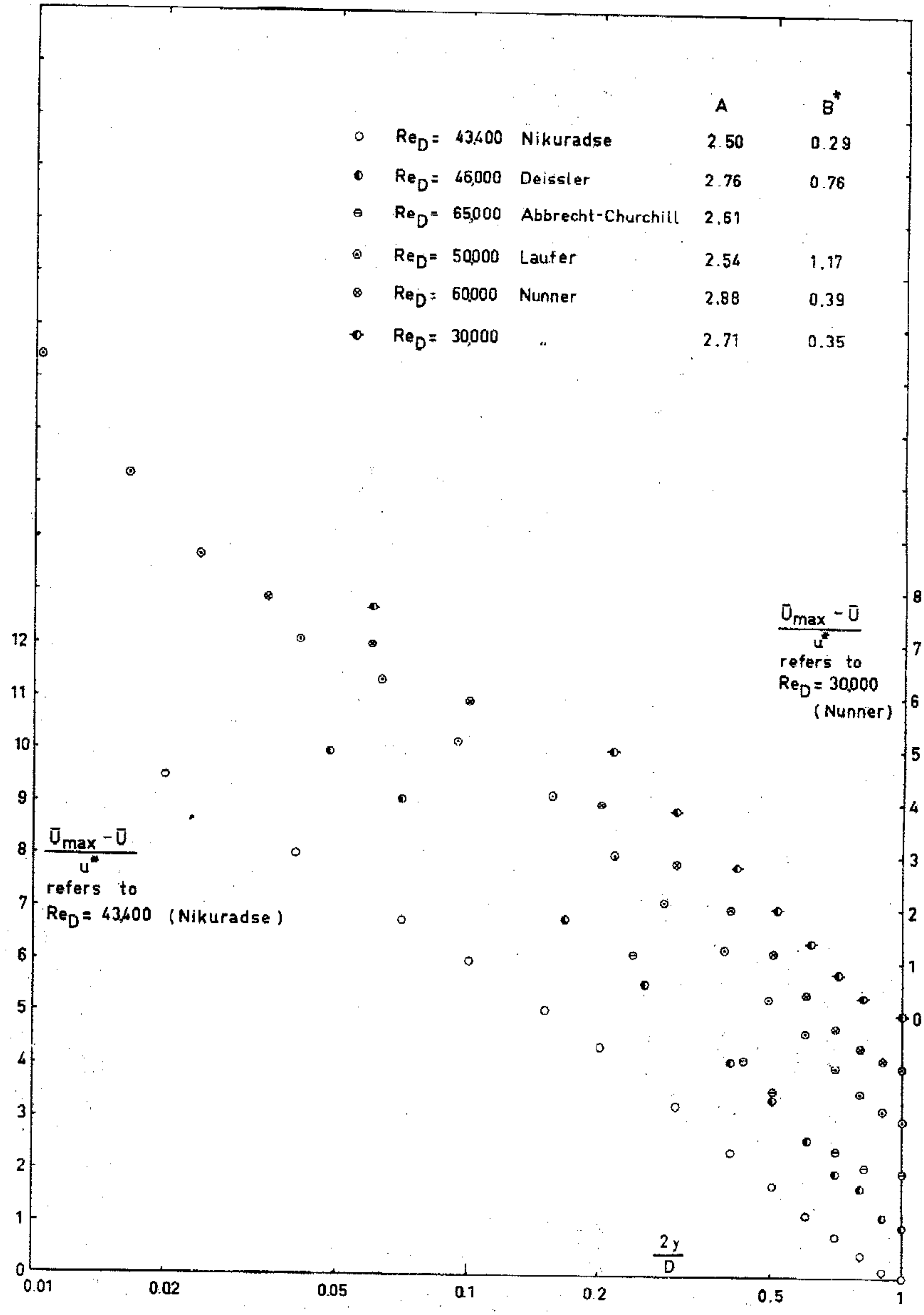

FIGURE 13
Velocity-defect distribution measured by various investigators.

Nikuradse used six different sizes of sand grains, so that it was possible to vary the relative roughness $\frac{D}{2k}$ between 15 and 507. Figure 14 shows the velocity distribution in the wall region for the two roughnesses $\frac{D}{2k} = 15$ and 252 and at various Re_D. The data referring to the points with a distance less than the total-head tube diameter from the wall have been omitted. Figure 15 gives for the same two roughnesses the velocity defect. If in the semi-logarithmic plot again a straight line is drawn through the points of the overlapping region, it appears that here too the constant A shows a similar dependence on Re_D, though less pronounced, than in the case of a smooth wall. The variation of the constants A, B′ and B* with Re_D is shown in Fig. 16.

Hitherto we have mainly considered the turbulent overlapping part of the wall region, whose mean-velocity distribution may be approximated by a logarithmic distribution. Outside the wall region, i. e. in the core region deviations from the logarithmic distribution occur. Of course it is possible, as suggested by Millikan, to introduce a correction function which in the velocity-defect representation should be a universal function of the relative distance $\frac{2r}{D}$ if the Reynolds number similarity would apply. Induced by the almost parabolic shape of the velocity-defect distribution in the core region it has also been suggested to approximate this distribution by the following function

$$\frac{\overline{U}_{max} - \overline{U}}{u^*} = K\left(\frac{2r}{D}\right)^n \tag{18}$$

In the case of Reynolds number similarity n should be independent of Re_D.

A purely parabolic distribution is obtained for $n = 2$ which would imply a constant ratio between the local shear stress and the mean-velocity gradient (constant eddy viscosity).

The function (18) has been applied to the velocity distributions measured by the mentioned investigators. Figures 17 and 18 show the results for a smooth pipe and Fig. 19 for a rough pipe. From these figures it is concluded that in general only a part of the velocity-defect distribution in the core region may be approximated by the function (18). Deviations not only occur towards the wall region (as it should be), but also in the region at the centre of the pipe. Furthermore the exponent n is only close to the value of 2 in Reichardt's and Page's results obtained in a channelflow. In general it is much less than 2 and shows some variation with Re_D. This latter is shown in Fig. 20.

The eddy-viscosity concept

The logarithmic velocity-distribution has been obtained from general arguments concerning the law of the wall and the velocity-defect law. But as known, it can be obtained also direct from an integration of the equation of motion, if the turbulence shear stress is assumed to be directly proportional to the local mean-velocity gradient, and that the coefficient of proportionally increases linearly with the distance y to the wall. This coefficient of proportionality is usually referred to as the eddy-viscosity ϵ of the flow. The logarithmic velocity-distribution is obtained if it is assumed that

$\tau = \rho \epsilon \frac{du}{dy}$

$\tau \sim \frac{du}{dy}$

$\epsilon \sim y$

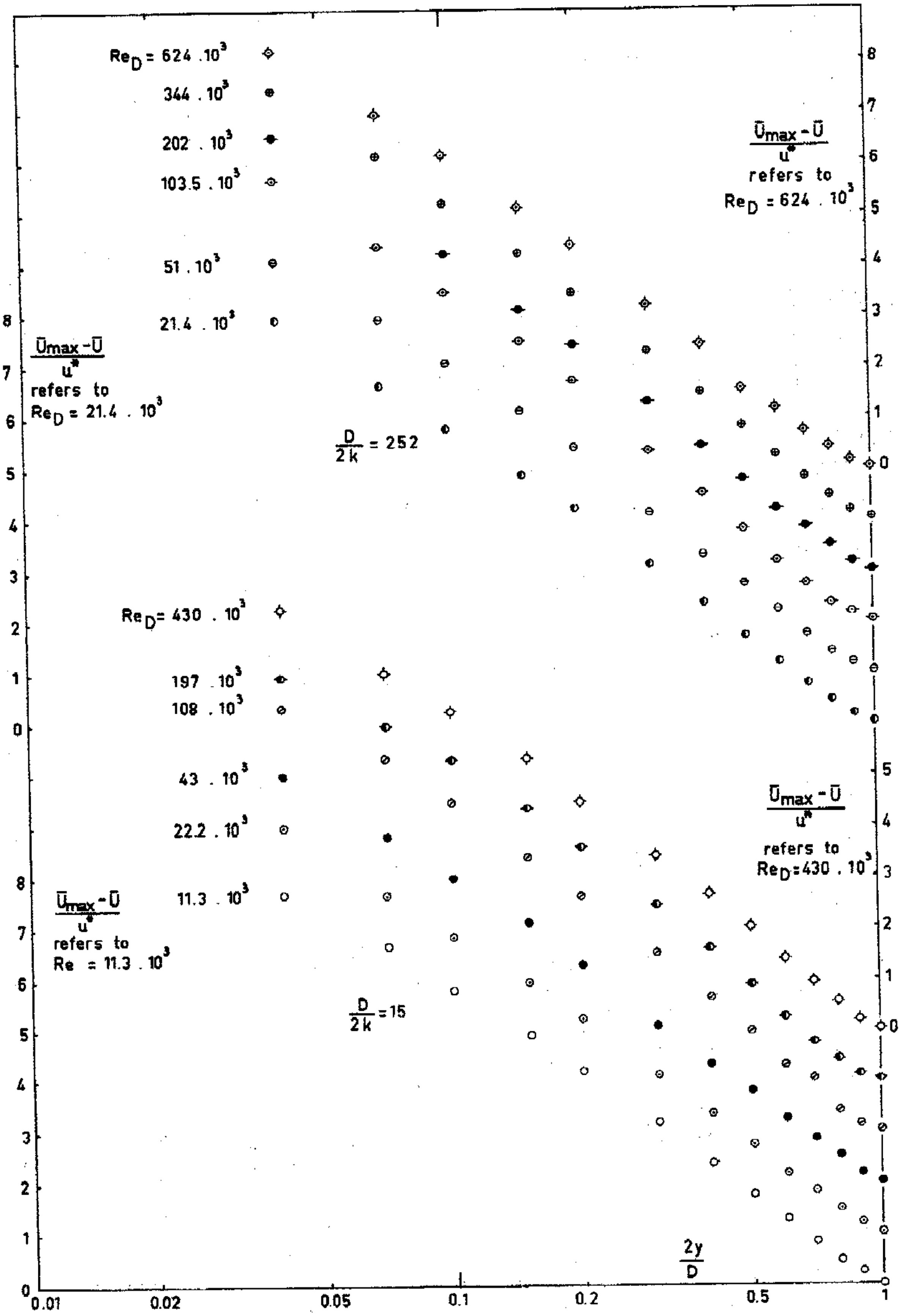

Figure 15
Velocity-defect distribution for rough pipes according to Nikuradse.

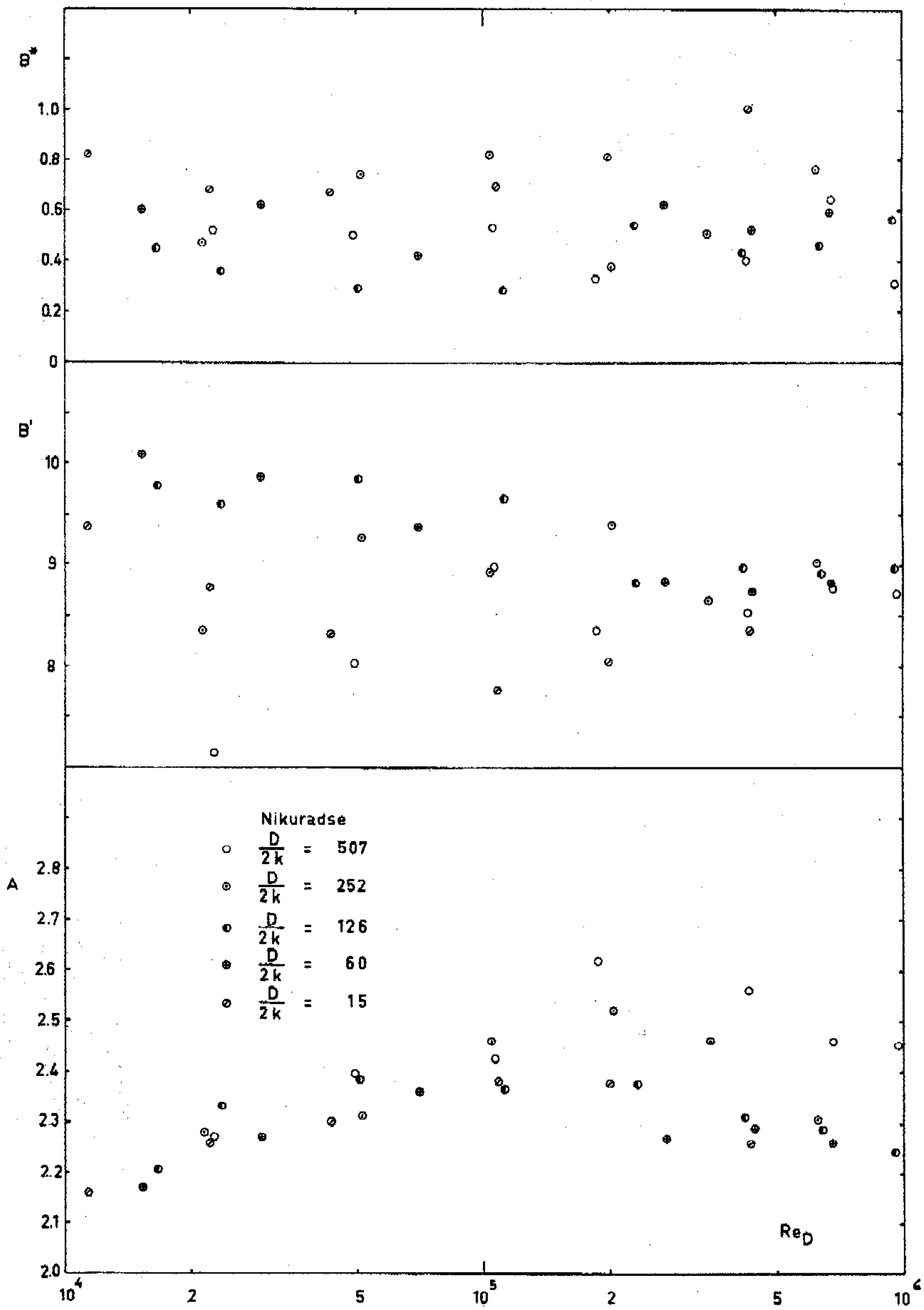

FIGURE 16
Effect of Re_D on A, B' and B* for rough pipes.

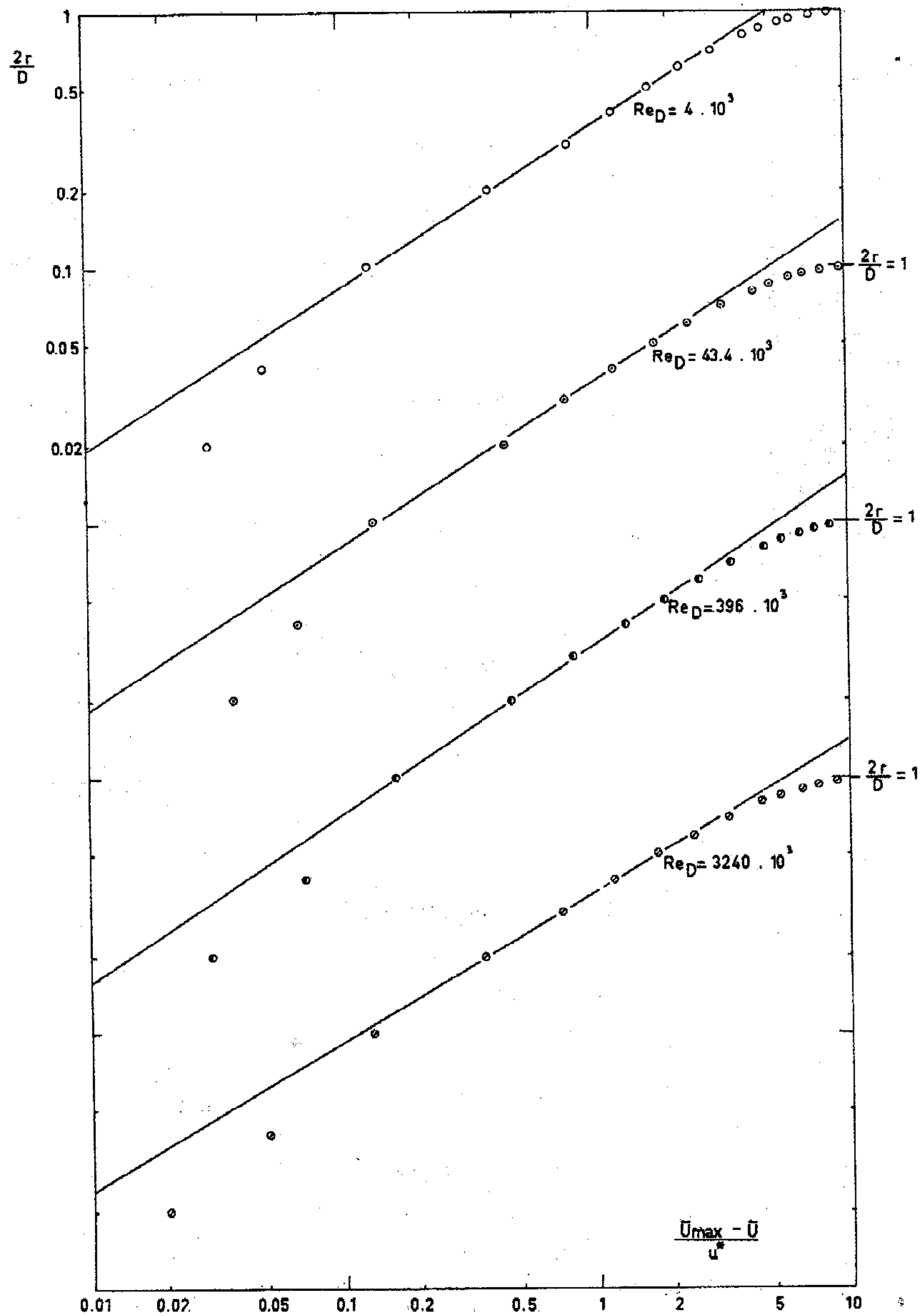

FIGURE 17
Velocity-defect in the core region of a smooth tube (Nikuradse).

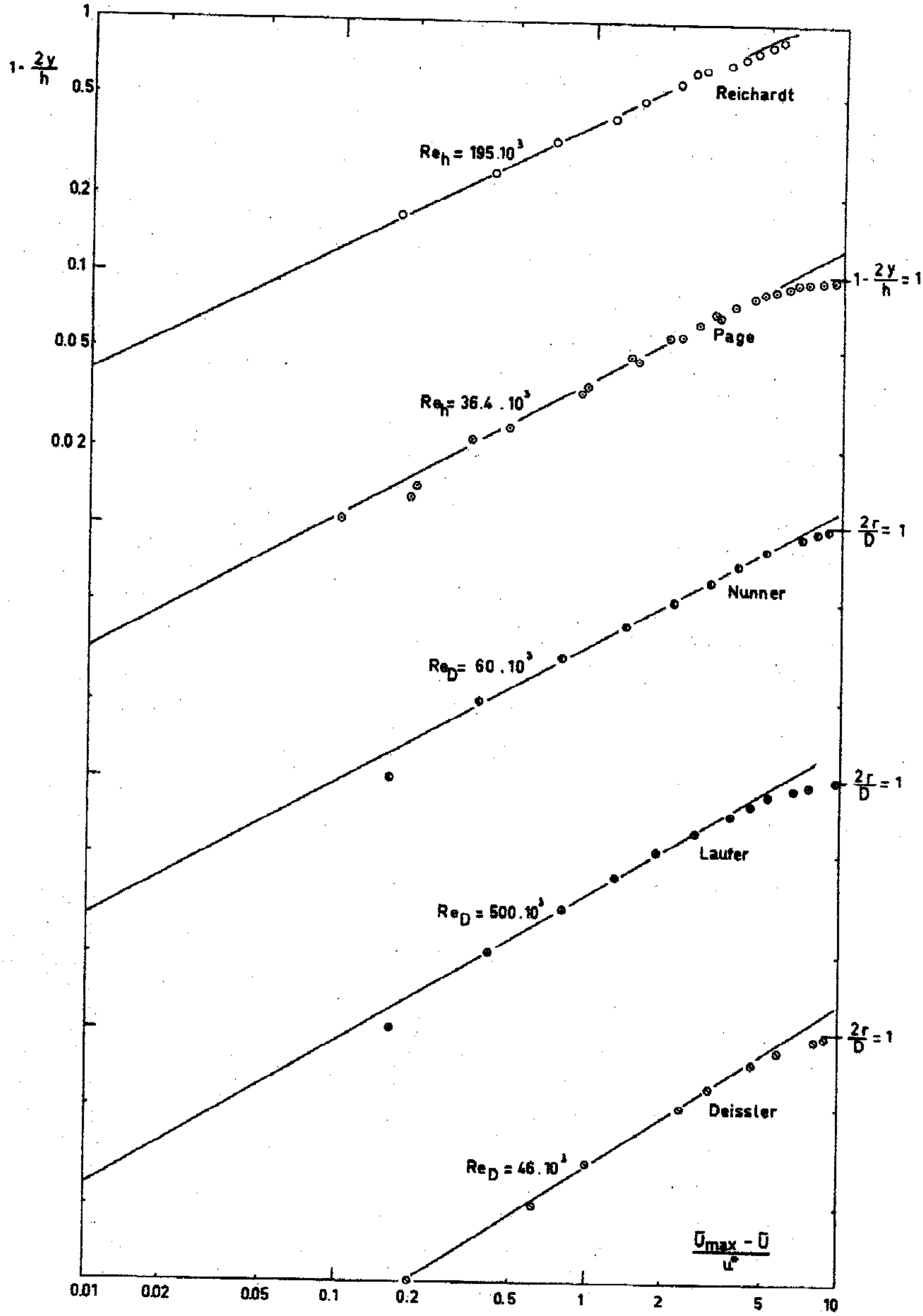

FIGURE 18
Velocity-defect in the core region of a smooth tube.

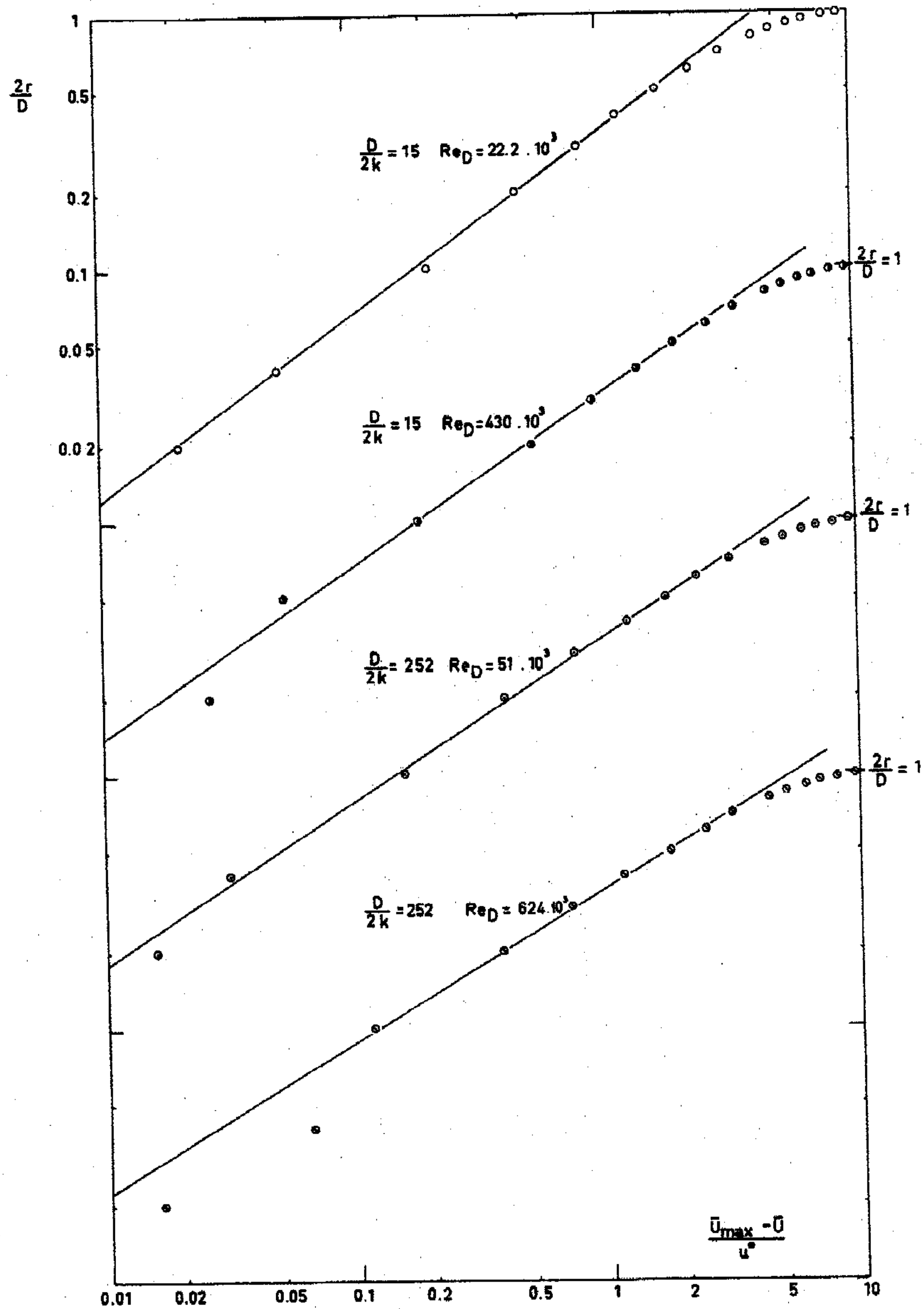

FIGURE 19
Velocity-defect in the core region of a rough tube (Nikuradse).

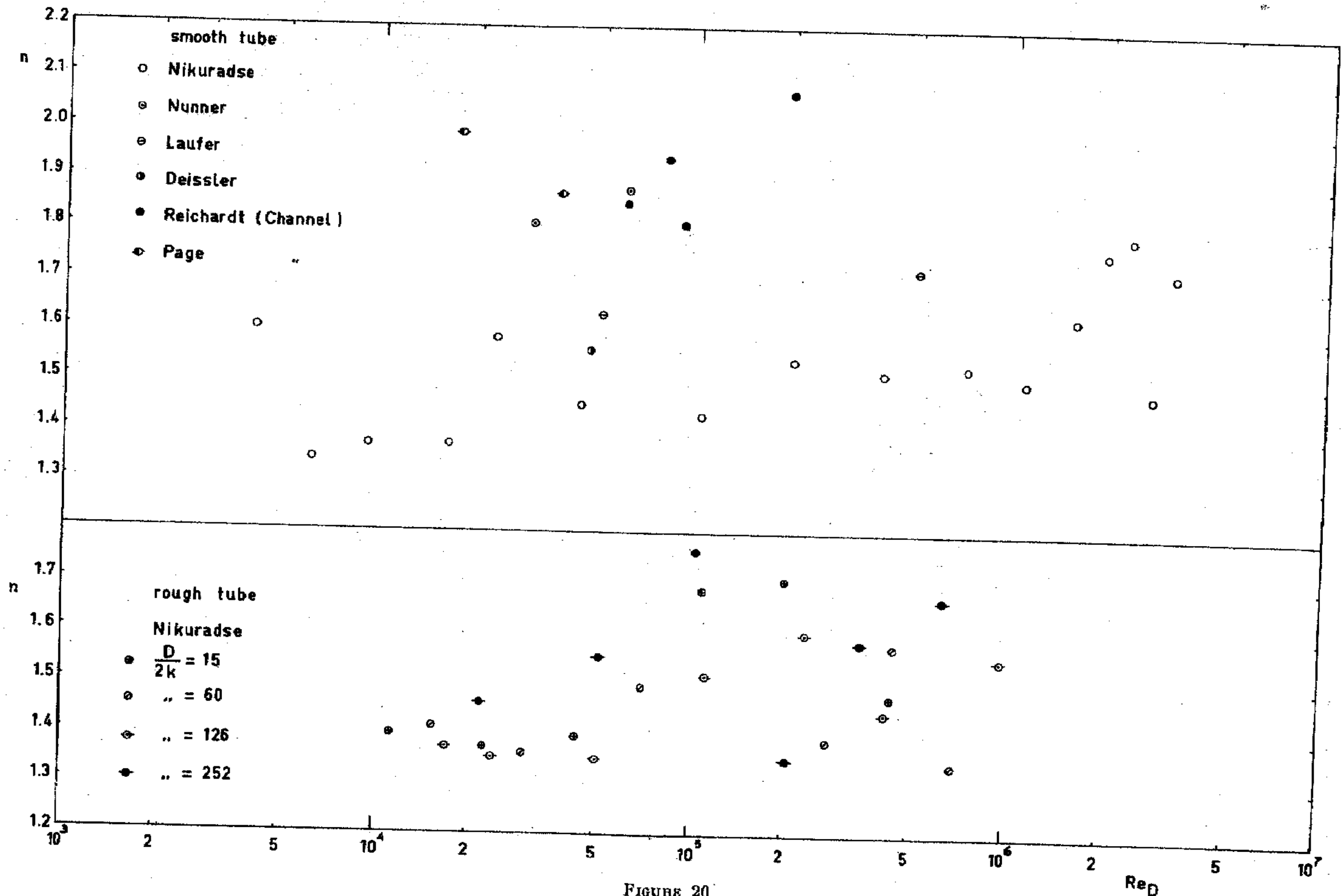

FIGURE 20
Effect of Reynolds number on the exponent n.

$$\epsilon = \frac{1}{A} u^* y \tag{19}$$

Also the velocity-defect distribution given by the function (18) is obtained if a suitable variation of ϵ with r in the core region is assumed, namely

$$\epsilon = \frac{u^* D}{n K} \left(\frac{2r}{D} \right)^{2-n} \tag{20}$$

In principle the introduction of an eddy-viscosity may not be correct, for it is not certain that the turbulence shear-stress $\rho \overline{u_r u_x}$ can be expressed in terms of purely local quantities, such as the local mean-velocity gradient (see e. g. ref. 11 and 12, chapter 5). For if the eddy-viscosity is written $\epsilon = u' l$, for a gradient-type of transfer the length l should be so small that the mean-velocity gradient may be considered as almost constant in a region of the dimension l; this means for instance that l should be small compared with Von Kármán's mixing length divided by the Von Kármán's constant. In turbulent pipe flow u' may be taken of the magnitude u^*. With the current values for ϵ in pipe flow it then turns out that l achieves values which do not or hardly satisfy the condition for a gradient type transfer, in particular in the wall region and close to the centre of the pipe.

If in the region in the vicinity of a smooth wall u^* and $\frac{u^*}{\nu}$ are the only parameters describing the flow, $\frac{\epsilon}{\nu}$ should be a universal function of $\frac{u^* y}{\nu}$. Similarly if Reynolds number similarity holds for the core region, $\frac{\epsilon}{u^* D}$ should be a universal function of $\frac{2r}{D}$. Since experimental evidence seems to indicate that in the turbulent part of the wall region and in the core region the values of A and n respectively are not independent of Re_D also $\frac{\epsilon}{\nu}$ and $\frac{\epsilon}{u^* D}$ must be dependent still on Re_D.

Since the values of ϵ are obtained from a differentiation process applied to the mean-velocity distribution, which procedure is pretty inaccurate, these values cannot be very reliable. Moreover the velocity distributions as measured under the same flow conditions (Re_D) by different investigators in particular in the core region, still show important differences. See for instance Figure 17 and 18, and the appreciable variations at the same Re_D in the value of n shown in Fig. 20.

NIKURADSE has determined the values of ϵ for the turbulent region of the pipe flow not from a graphical differentiation process of the mean-velocity distribution, but from values of the local mean-velocity gradient measured directly with two tiny total-head tubes placed close to each other. His results show that with increasing distance from the wall ϵ first increases almost linearly, obtains a maximum value at $\frac{2y}{D} \simeq 0.5$ and decreases with further increasing distance to a sharp minimum at the centre of the pipe. There is still a dependence of $\frac{\epsilon}{u^* D}$ on the Reynolds number, it decreases slightly with increasing Re_D. Figure 21 gives a plot of $\frac{2\epsilon}{D}$ against Re_D with the relative distance

$\frac{2r}{D}$ as a parameter, where this decrease with increasing Re_D is clearly shown. In the core-region, however, $\frac{2\epsilon}{u^* D}$ approaches a constant value at very high Reynolds numbers ($Re_D > 10^5$ to 10^6), but as the wall is approached there is still a decrease even at the highest values of Re_D.

The same trend is observed for the flow through a rough pipe. Figure 22 shows this trend for two relative roughnesses $\frac{D}{2k} = 15$ and 252. There appears to be still a viscosity effect, even at conditions which are usually referred to as hydraulically rough. At this condition the friction coefficient is assumed to be independent of the Reynolds number. But, as may be expected, the friction coefficient must be much less sensitive to the additional viscosity effect as observed on the eddy viscosity and on the mean velocity distribution.

If the wall is approached the eddy-viscosity decreases rapidly to zero, in the turbulent region linearly with the distance to the wall, but in the transition region and viscous sublayer at a much greater rate. Figure 23 shows the variation of $\frac{\epsilon}{\nu}$ with $\frac{u^* y}{\nu}$ in the region close to the wall, as obtained by ABBRECHT and CHURCHILL from their mean-velocity distribution measurements at a $Re_D \simeq 65.10^3$. From the mean-velocity distribution in the turbulent overlapping region $A = 2.61$ is obtained, so that the linear part of the $\frac{\epsilon}{\nu}$ curve is given by the equation $\frac{\epsilon}{\nu} = 0.38 \frac{u^* y}{\nu}$. In the region $\frac{u^* y}{\nu} < 30$ $\frac{\epsilon}{\nu}$ seems to follow a quadratic relation with $\frac{u^* y}{\nu}$, but there are theoretical reasons to believe that the decrease with decreasing distance must be at a still higher rate as $y \to 0$ (see e.g. ref. 4). If the fluid is incompressible a series expansion of the axial and radial turbulence velocity components u_x and u_r yields

$$-\overline{u_x u_r} = a_3 y^3 + a_4 y^4 + \ldots$$

where

$$a_3 = \frac{1}{2}\left[\overline{\left(\frac{\partial u_x}{\partial y}\right)_0 \frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial y}\right)_0} + \overline{\left(\frac{\partial u_x}{\partial y}\right)_0 \frac{\partial}{\partial z}\left(\frac{\partial u_\phi}{\partial y}\right)_0}\right]$$

and

$$a_4 = \frac{1}{4}\left[\overline{\left(\frac{\partial^2 u_x}{\partial y^2}\right)_0 \frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial y}\right)_0} + \overline{\left(\frac{\partial^2 u_x}{\partial y^2}\right)_0 \frac{\partial}{\partial z}\left(\frac{\partial u_\phi}{\partial y}\right)_0}\right] + \frac{1}{6}\left[\overline{\left(\frac{\partial u_x}{\partial y}\right)_0 \frac{\partial}{\partial x}\left(\frac{\partial^2 u_x}{\partial y^2}\right)_0} + \overline{\left(\frac{\partial u_x}{\partial y}\right)_0 \frac{\partial}{\partial z}\left(\frac{\partial^2 u_\phi}{\partial y^2}\right)_0}\right]$$

So the turbulence shear-stress $-\rho \overline{u_x u_r}$ and consequently the eddy viscosity $\frac{\epsilon}{\nu}$ decreases at least with y^3 as $y \to 0$. In steady pipe flow which is homogeneous in axial direction the first term in the expression for a_3 is zero, but the second term need not be necessarily zero. The available experimental evidence are entirely insufficient to give any decision as to the actual behaviour of $\frac{\epsilon}{\nu}$ very close to the wall.

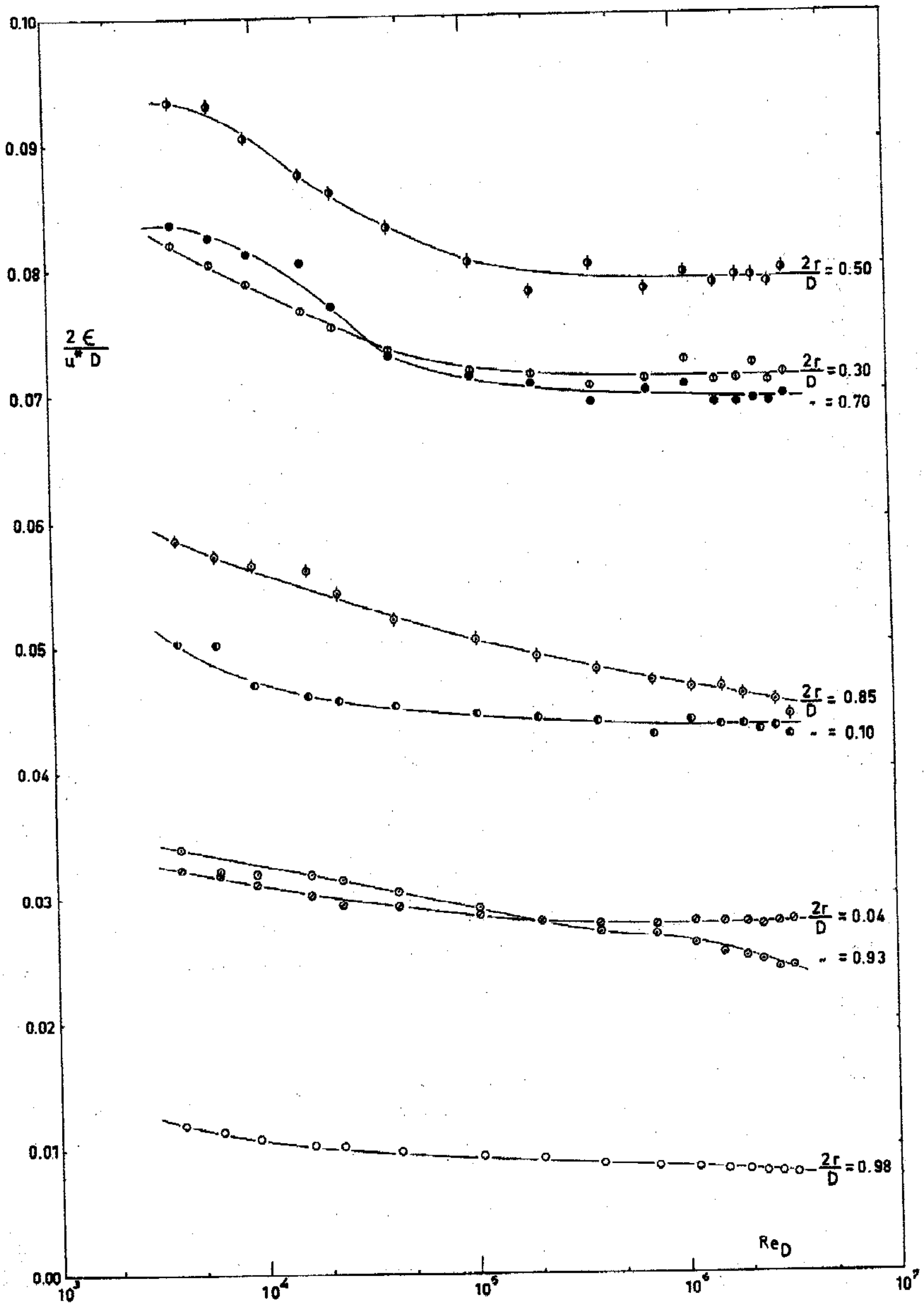

FIGURE 21
Effect of Reynolds number on the eddy viscosity. Smooth-wall condition (Nikuradse).

The above results concerning the effect of Re_D on $\frac{\epsilon}{\nu}$ or $\frac{\epsilon}{u^* D}$ are as matter of fact not so very surprising, since the detailed turbulence structure is strongly affected by viscosity. The data on the turbulence structure in pipe flow seem to indicate that the relative turbulence intensities $\frac{u'}{u^*}$ etc. are still, slight, functions of Re_D. Similarly the integral scales of turbulence seem to decrease with increasing Re_D. Though it must be admitted immediately that the number of available data is still very small and the data are not very accurate.

The roughness problem

A few words may be devoted to the problems which one has to face if the turbulent flow along a rough wall has to be described. In the section on the mean-velocity distribution, when discussing NIKURADSE's experiments with rough tubes one problem has been mentioned already, namely to establish the effective origin of the mean-velocity distribution. The way in which NIKURADSE has « solved » this problem need not be the correct one. From mean-velocity measurements in the flow past a wall on which were fitted hemi-spherical elements EINSTEIN [13] concluded that the effective origin should be taken at 0.4 k below the top of the elements, in order to obtain a satisfactory straight portion in the semi-logarithmic plot direct beyond the roughness elements. In general this effective origin may be found by trial and error by replotting the mean-velocity distribution with various positions of the origin till a satisfactory straight portion in the above plot is obtained.

Another, more complex problem is how to take into account the effect of non-uniform roughnes and different types of roughnesses. It is well-known from measured friction coefficients that if the roughness is of a wavy nature with an average amplitude small compared with the average wavelength there is a pronounced Reynolds number effect almost similar to the effect with a smooth wall, and quite different from the effect of sand roughness. Also the shape, configuration and size distribution of the roughness elements have been found to have a marked effect on the flow. It is obvious that one single roughness parameter k must be insufficient to describe the effect. This effect can be demonstrated clearly as follows.

If for the moment we assume that the Reynolds number similarity and the law of the wall still hold to a sufficient degree of approximation for the following consideration then from the velocity-defect law it is concluded that the effect of the wall roughness has to be found in the value of $\frac{\overline{U}_{max}}{u^*}$, which may differ from the smooth wall condition by a value $\frac{\Delta\overline{U}}{u^*}$. Also, as HAMA [14] has shown, the velocity distribution in the turbulent part of the wall-region may be written

$$\frac{\overline{U}}{u^*} = A \ ln \frac{u^* y}{\nu} + B - \frac{\Delta\overline{U}}{u^*}$$

where

$$\frac{\Delta\overline{U}}{u^*} = A \ ln \frac{u^* k}{\nu} + B - B'$$

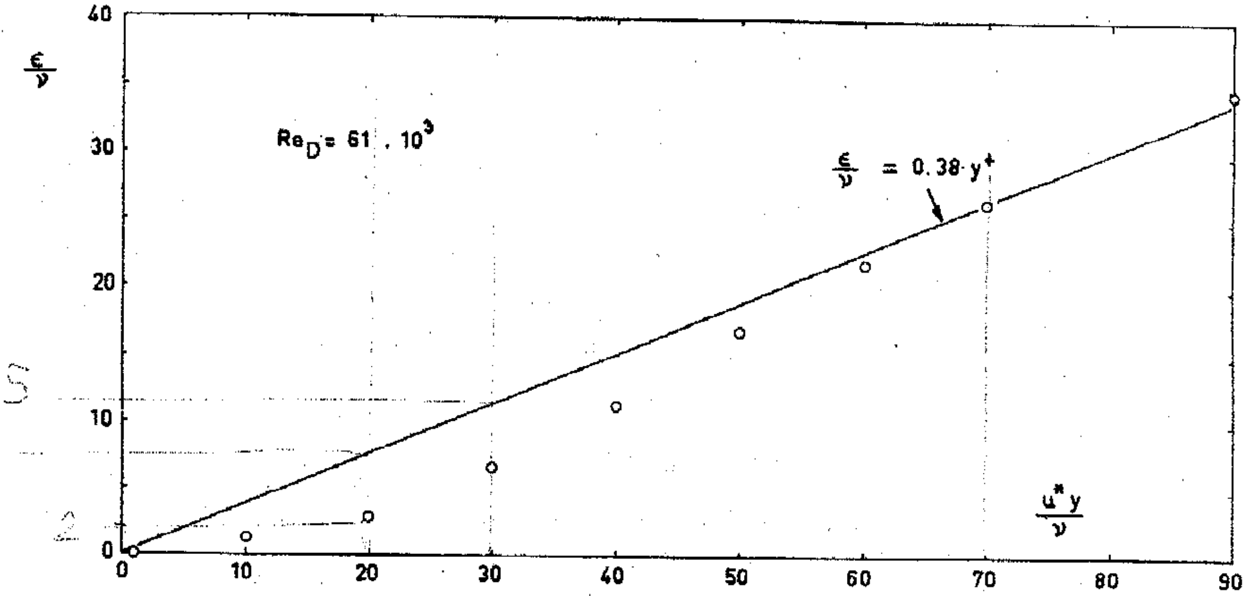

FIGURE 23
Variation of the eddy viscosity in the wall region close to the wall (Abbrecht and Churchill).

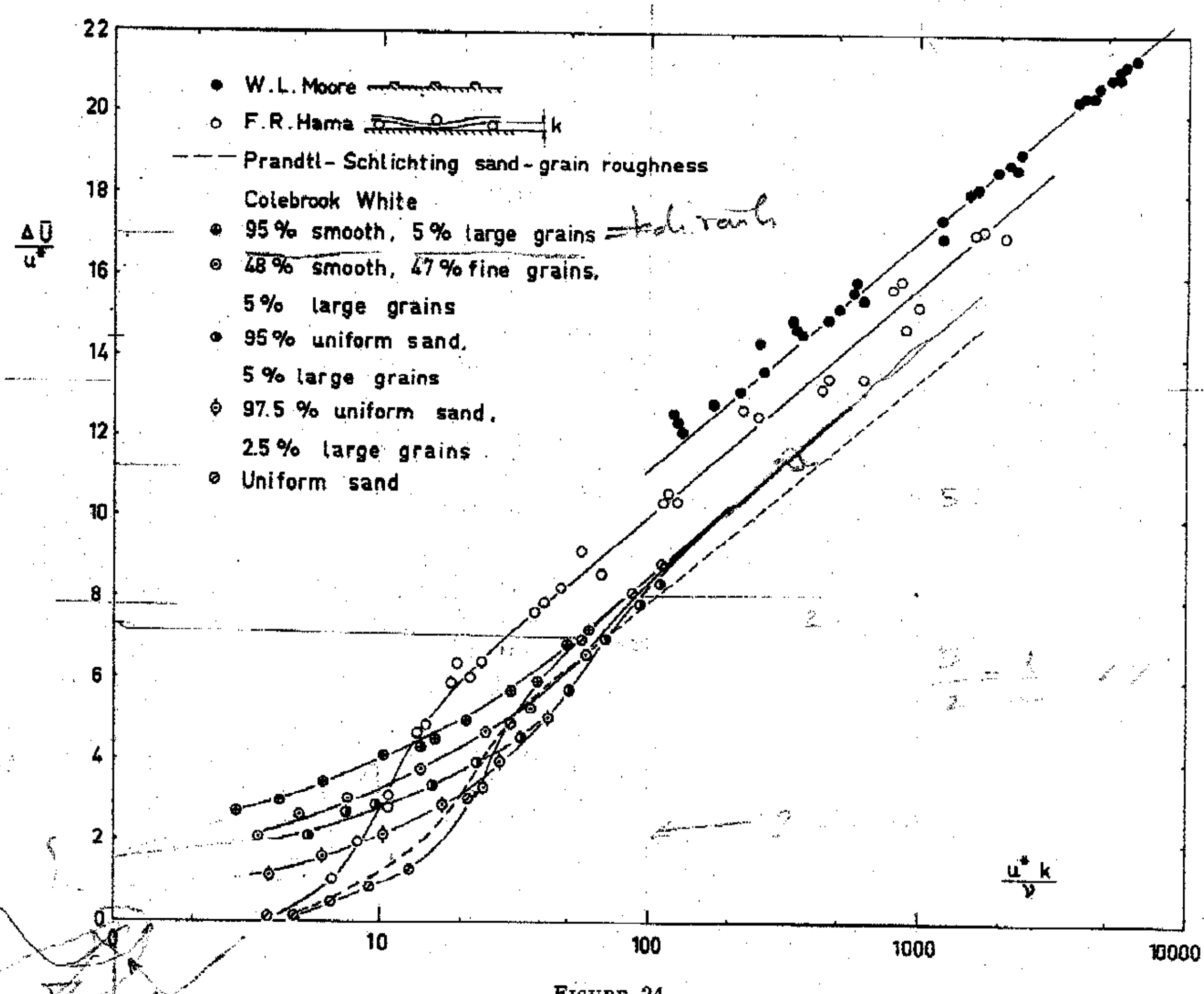

FIGURE 24
Effect of wall roughness on the shift of the velocity distribution profile.

Now if the parameter k would be sufficient to account for the observed effect, $\frac{\Delta \overline{U}}{u^*}$ should be a universal function of k. Figure 24 reproduced from CLAUSER's [15] publication, shows $\frac{\Delta \overline{U}}{u^*}$ as a function of $\frac{u^* k}{\nu}$ for various types of roughnesses. From this figure it is concluded that the shift $\frac{\Delta \overline{U}}{u^*}$ is only proportional to $ln \left(\frac{u^* k}{\nu}\right)$ with a constant of proportionality equal to A at sufficient large values of $\frac{u^* k}{\nu}$ (hydraulic rough condition), but that $B - B'$ depends on the type of roughness. The value of $\frac{u^* k}{\nu}$ beyond which the wall may be considered as hydraulic rough also depends on the type of roughness.

Figure 25, again shows $\frac{\Delta \overline{U}}{u^*}$ for sand roughness according to NIKURADSE's measurements, and a few data obtained from NUNNER's publication for a certain type of artificial roughness. These latter data demonstrate for the same roughness elements the important effect of the geometrical configuration.

Hitherto the difficulty in obtaining the correct parameter(s) to account for the effect of a certain type of roughness has been got round by the introduction of the equivalent sand roughness k_e, which is the sand roughness which produces at the same Reynolds number Re_D the same friction coefficient as the actual roughness. In principle it should be always possible to obtain such a value of k_e for any type of roughness, but it has only practical value if this equivalent sand roughness for a certain type of roughness is independant of $\frac{u^* k}{\nu}$. Figures 24 and 25 show this is not always possible, especially if the wall condition is not hydraulic rough, so that the direct viscosity effect is still appreciable.

TOWNSEND [10] has suggested to introduce another roughness factor k_0 which makes that the logarithmic velocity distribution for a hydraulic rough condition of the wall (16) contains the same value for the constant B′ as for smooth wall

$$\frac{\overline{U}}{u^*} = A \; ln \frac{y}{k_0} + B$$

The relation between k_0 and k is

$$A \; ln \frac{k_0}{k} = B - B'$$

But in principle the objections made against the use of an equivalent sand roughness also apply here.

Not yet satisfactorily solved is the case of composite roughness. A simple example of it is found when the wall is « wavy » on which is super-imposed an irregular roughness of a much finer scale. If the difference between the scale of the wavy roughness and the irregular roughness is large, the flow may be considered as to take place along a wall of the fine irregular roughness, but which is undulating instead of plane. So the axial mean-pressure gradient is not uniform in flow direction.

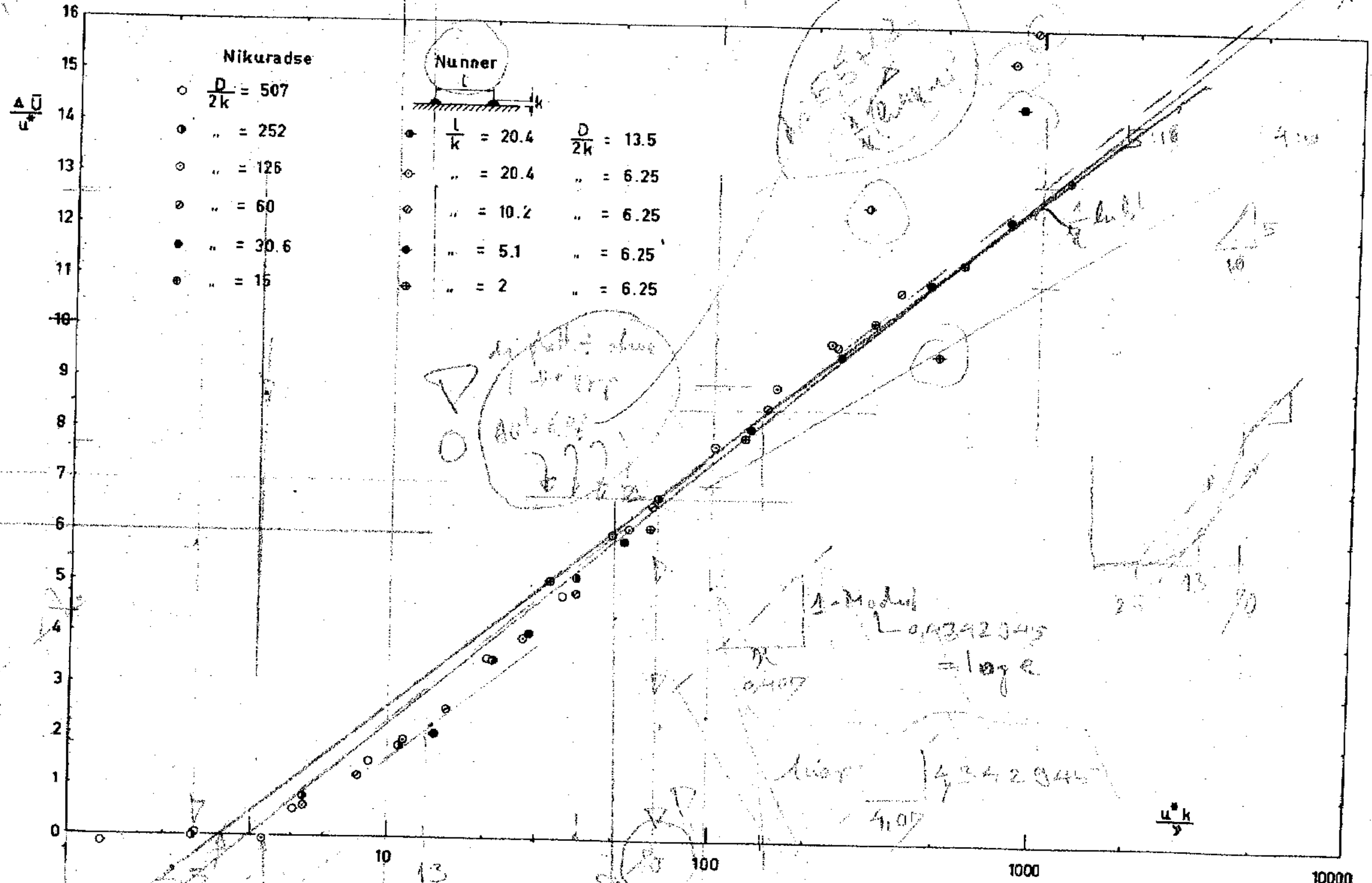

FIGURE 25

Effect of sand roughness and artificial roughness on the shift $\frac{\Delta \overline{U}}{u^*}$ of the velocity distribution.

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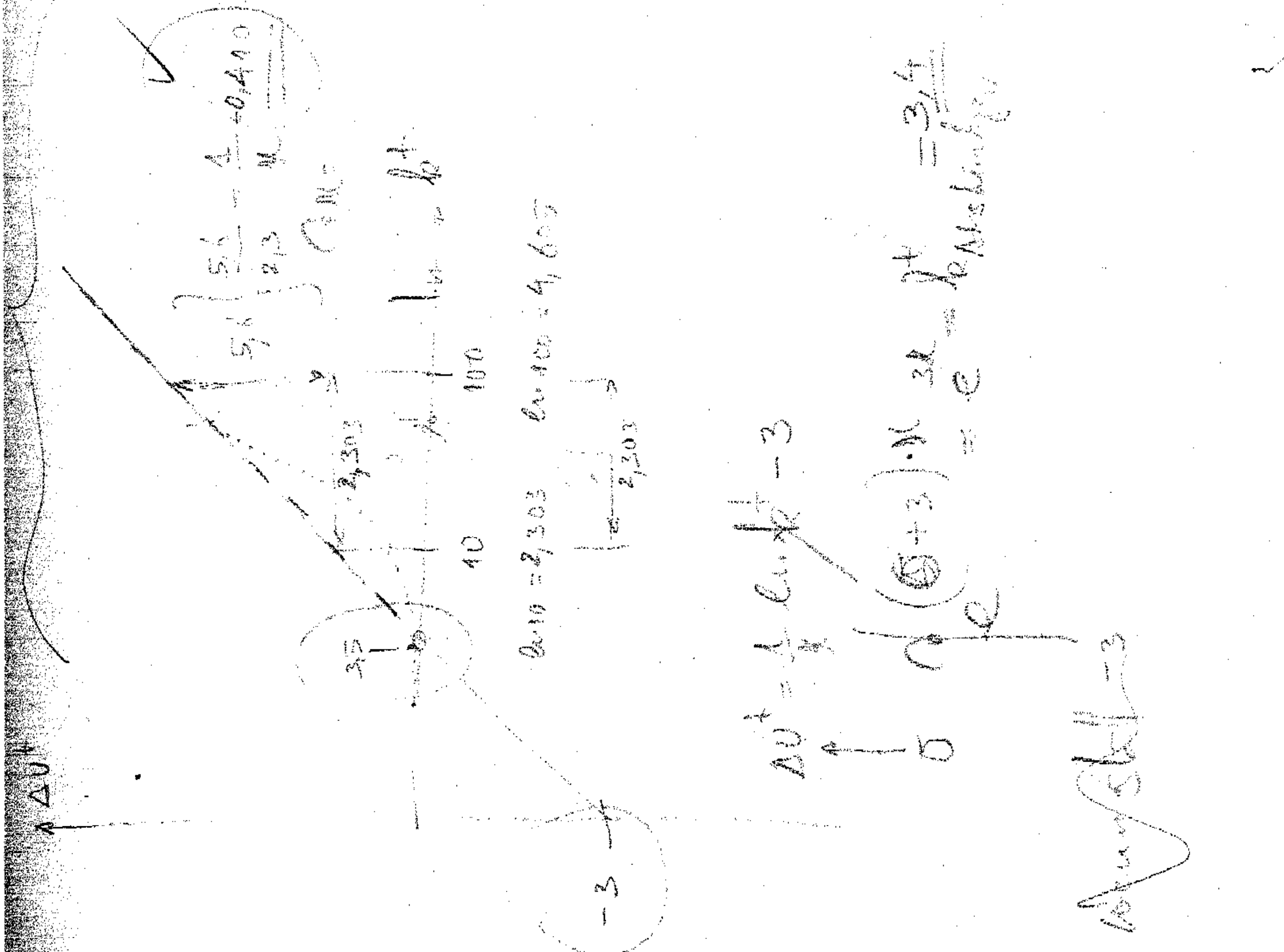

REMARKS ON THE MALKUS THEORY OF TURBULENT FLOW

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SOMMAIRE

Le but de cette revue est :

1) De présenter les grandes lignes de la théorie de la turbulence de Malkus, ainsi que les faits expérimentaux qui semblent montrer que cette théorie donne une meilleure description des écoulements entre parois planes et parallèles, que les théories phénoménologiques usuelles.

2) De considérer la façon dont les hypothèses de cette théorie sont reliées aux connaissances acquises des structures turbulentes, fondées sur les mesures, et traduites sous forme de concepts à partir de la théorie statistique de la turbulence.

3) D'insister sur certaines des difficultés qui surgissent lors de l'application de ces concepts à des écoulements turbulents en cours de développement.

SUMMARY

The purpose of this review is (i) to present an outline of the Malkus theory of turbulent flow and the experimental evidence that seems to show that the theory gives a better description of flows between parallel plane surfaces than current phenomenological theories, (ii) to consider the way in which the assumptions of the theory are related to existing knowledge of turbulent structure based on measurement and interpreted in terms of concepts from the statistical theory of turbulence, and (iii) to point out some of the difficulties that arise in the application of the concepts to developing turbulent flows.

1. Introduction

Since 1935 when G. I. TAYLOR initiated the statistical theory of turbulence, nearly all work on the fundamental nature of turbulent flow has been strongly influenced by this theory and by the hope that some day it would develop into an analytical theory derivable from the equations of motion and capable of describing shear flows without arbitrary assumptions. A considerable amount of effort on these lines has left us with a detailed experimental knowledge of a variety of flows and a body of theory which applies mostly to homogeneous turbulence and which depends on assumptions that are arbitrary and, in their own way, as open to question as the assumptions of mixing-length theory. Moreover, the extension of this theory to inhomogeneous shear-flows presents difficulties that can only be overcome by making additional and far more dubious assumptions, so that most current theories of shear-flow are frankly phenomenological and differ only in the amount of attention paid to the fluctuating motion. In spite of its failure to realise our hopes, the statistical theory continues to provide the framework and concepts for our understanding of the basic processes in turbulent flow, and this makes it difficult to appreciate fully a theory of turbulent flow based on quite different concepts. The theory of turbulent flow put forward by W. V. R. MALKUS in 1954 rejects two characteristic features of the statistical theory, the detailed satisfaction of the equations of motion and the use of mean values other than those describing the macroscopic transport, and introduces a number of new assumptions of a thermodynamic rather than a mechanical nature. This novelty and the complexity of the associated mathematics has made for slow appreciation of the merits of the theory, but the evidence that has accumulated to show that its predictions describe the transport of heat and momentum between parallel planes better than do the phenomenological theories calls for further study and a little bridge-building. The purpose of this review is to discuss the basic assumptions of the theory and their justification, the experimental observations that confirm its predictions, the relation of its concepts to the concepts of the statistical theory and the possibility of describing developing flows by a modified form of the theory.

2. Notation

The flows considered are between parallel, horizontal plane surfaces, and are described in a co-ordinate system with Oz vertically upwards and Ox in the direction of mean flow. The bounding surfaces are $z = 0$, $z = D$, and position in the flow is specified by the non-dimensional variable, $\Phi = \frac{\pi z}{D}$ with a range of $0 - \pi$. Pressures and stresses are « kinematic », i. e. their mechanical values divided by the fluid density. The fluid is considered to be incompressible and it is assumed that effects of gravity are limited to variable weight. Then,

$U + u, v, w$	are the instantaneous components of velocity;
U	is the mean velocity;
$T + \theta$	is the instantaneous, absolute temperature;
T	is the mean temperature;
P	is the mean pressure;
$\tau_0 = -\frac{1}{2} D \frac{dP}{dx}$	is the shear-stress on the lower boundary;
$H = \rho c_p T Q$	is the upward flux of total heat (enthalpy);
ν	is the kinematic viscosity;
κ	is the thermometric conductivity;
T_a	is the mean absolute temperature at $z = \frac{1}{2} D$;
T_1, T_2	are the temperatures of the lower and upper surfaces.

3. The basic assumptions of the theory

In its original form (Malkus, 1954*b*, 1956), the theory is applicable to stationary flows, statistically homogeneous on planes parallel to the bounding surfaces, and he has carried out the necessary calculations for Poiseuille flow between parallel, plane boundaries and for heat convection between horizontal, parallel planes. Consider the Poiseuille flow first. The rate of turbulent transport of momentum can be expressed as an infinite series of orthogonal functions, for example as a Fourier series,

$$\overline{uw} = \sum_{1}^{\infty} b_\tau \sin 2\tau\Phi \tag{3.1}$$

and the first assumption is that a satisfactory description is obtained by terminating this series after n_0 terms, the value of n_0 depending on the Reynolds number of the flow. This assumption is physically consistent with the restrictions imposed on u and w by the presence of the boundaries only if each one of the set of orthogonal functions satisfy these restrictions. For solid boundaries, circular functions do not satisfy this condition but the proper orthogonal functions resemble closely circular functions except within a wave-length of the boundaries and, for the present purpose*, the difference may be neglected. Since the averaged equation of motion for Poiseuille flow is

$$\frac{d\,\overline{u\,w}}{dz} = \frac{2\,\tau_0}{2\mathrm{D}} + \nu\,\frac{d^2\,\mathrm{U}}{dz^2}$$

equation (3.1) may be written in the form,

$$\frac{d^2\,\mathrm{U}}{d\Phi^2} = \sum_{0}^{n_0} a_\tau \cos 2\tau\Phi \tag{3.2}$$

where

$$a_0 = \frac{\tau_0\,\mathrm{D}}{2\,\pi^2\,\nu}$$

The values of n_0 and the remaining a_r's are then determined by requiring that, for a given total flow, the energy dissipation of the flow is to be a maximum with respect to variations of n_0 and the a_r's such that

$$\frac{d^2\,\mathrm{U}}{d\Phi^2} \leqslant 0 \quad (0 < \Phi < \pi) \tag{3.3}$$

and that an infinitesimal disturbance of the form corresponding to the last term is neutrally stable in a laminar flow with the same distribution of mean velocity.

When these conditions are satisfied, non-linear interactions between the modes of motion represented by terms in the series (3.2) and other turbulent motions not transferring momentum are supposed to be sufficient to keep the mode amplitudes stationary in time, that is to say, non-linear transfer acts as a stabilising influence. This influence is negligible on the mode of highest order which is neutrally stable in consequence of

* i.e. the determination of the velocity profile outside the viscous layers.

effects considered by ordinary stability theory of infinitesimal disturbances, i. e. a balance between energy-transfer from the mean flow and energy-loss by direct viscous dissipation. These effects of non-linear energy transfer should be distinguished from the energy-transfer between eddy components as considered in the statistical theory (see for example BATCHELOR, 1953). The modes of motion considered by Malkus are motions obtaining energy directly from the mean flow and losing it by non-linear transfer to a « background » of nearly isotropic eddies which contribute nothing to the momentum transfer. The background is not described by the theory in its simple form and includes those parts of the turbulent motion which are in a condition of local similarity.

The restriction $\frac{d^2 U}{d \Phi^2} \leqslant 0$ excludes points of inflexion in the velocity profile which lead to inviscid instability in the flow. This kind of instability is known to be much stronger than the viscous instability of flow without points of inflexion, and it might be expected that the appearance of these points would generate additional non-linear transfer tending to destroy them.

The requirement of maximum energy dissipation subject to the various restrictions is qualitatively reasonable but not the only plausible extreme condition. The qualitative justification is that, as the Reynolds number of the flow becomes very large, the velocity profile must become more and more flat-topped and tend to a constant value equal to the mean flow velocity with discontinuities at each wall. Since the total energy-dissipation is proportional to the velocity gradients at the walls, maximum dissipation corresponds to a best approach to the asymptotic shape for infinite Reynolds number. Other requirements can be found with the same asymptotic behaviour but their physical meaning is less clear.

It is natural to ask how much do the theoretical predictions, particularly those most easily tested by experiment, depend on validity of the four main major assumptions,

(i) Description of transport by a finite series,

(ii) Marginal stability of the last member of the series,

(iii) $\frac{d^2 U}{dz^2} \leqslant 0$,

(iv) Maximum energy-dissipation.

MALKUS (1960) has pointed out that the form of the velocity distribution for $n_0 \sin \Phi \gg 1$ is determined substantially by conditions (i) and (iii) alone if the coefficients a_r are distributed smoothly, i. e if $|a_r - a_{r+1}| \simeq \frac{a_1}{n_0}$ and he finds that maximising the dissipation with n_0 constant and then determining n_0 by condition (ii) gives very nearly the same result as maximising for variation of n_0 as well. This suggests that the form, though not the scale, of the velocity distribution is a consequence of termination of the series and the condition $\frac{d^2 U}{d \Phi^2} \leqslant 0$. This distribution can be obtained in an interesting way by considering the best way of approximating to the asymptotic distribution with a finite series giving non-positive values for $\frac{d^2 U}{d \Phi^2}$. Intuitively, we expect this result of values of $\frac{d^2 U}{d \Phi^2}$ as nearly zero as possible everywhere except very close to the walls if

$\frac{d^2 U}{d\Phi^2}$ takes the value zero for as many equally spaced values of Φ as possible, and all of these must be double zeros. Since

$$\sum_{0}^{n_0} a_1 \cos 2 r\Phi$$

can be expressed as a polynomial of degree $2n_0$ in $\cos \Phi$ (which is a single-valued function of Φ in the interval $0 - \pi$), n_0 double zeros are possible. The function

$$\left(\frac{\sin (n_0 + 1) \phi}{\sin \phi}\right)^2$$

may also be expressed as a polynomial of degree $2n_0$ in $\cos \Phi$, has n_0 double zeros equally-spaced at $\Phi_r = \frac{r\pi}{n_0 + 1}$ and so satisfies these conditions. Then

$$-\frac{d^2 U}{d\Phi^2} = A \frac{\sin^2 (n_0 + 1) \Phi}{\sin^2 \Phi} \tag{3.4}$$

where n_0 and the constant A are determined by condition (ii) and by the equation of mean motion*. If $n_0 \sin \Phi \gg 1$, the numerator oscillates rapidly with a mean value of $\frac{1}{2}$, and, if these oscillations are ignored,

$$-\frac{d^2 U}{d\Phi^2} = \frac{1}{2} A \operatorname{cosec}^2 \Phi$$

and

$$U_m - U = \frac{1}{2} A \log (\sin \Phi) \tag{3.5}$$

where U_m is the velocity at the channel centre. While the distribution (3.4) has the maximum number of equally-spaced zeros of $\frac{d^2 U}{d\Phi^2}$ for a given value of n_0, distributions of the form

$$-\frac{d^2 U}{d\phi^2} = A_1 \frac{\sin^2 (n_0 + 1) \phi}{\sin^2 \phi} + A_2 \frac{\sin^4 \left(\frac{n_0}{2} + 1\right) \phi}{\sin^4 \phi}$$

may also be written as a series

$$\sum_{0}^{n_0} a_1 \cos 2r\Phi$$

and approach the asymptotic distribution as n_0 becomes large. If the second therm is moderately small compared with the first, conditions other than maximum energy-

* Since circular functions have been used in the series (3.2) rather than orthogonal functions satisfying the boundary conditions, this expression is inaccurate close to the wall. Outside the viscous layers for $n_0 \sin \Phi \gg 1$, it is a good approximation to the result that would be obtained from the use of these orthogonal functions.

dissipation could be satisfied by choice of the coefficients. Ignoring the oscillations,

$$-\frac{d^2 U}{d\Phi^2} = \frac{1}{2} \mathbf{A}_1 \operatorname{cosec}^2 \Phi + \frac{3}{8} \mathbf{A}_2 \operatorname{cosec}^4 \Phi \tag{3.7}$$

for $n_0 \sin \Phi \gg 1$, and a comparison with (3.5) shows that the form of the velocity distribution is nearly unchanged since $\frac{A_2}{A_1} \simeq n_0^{-2}$ if the second term in (3.6) is to be nowhere dominant. This indicates that the exact nature of the extreme condition will not change the predicted distribution in form unless it departs considerably from the physically plausible condition of maximum dissipation.

Applying the condition of marginal stability of the highest mode to determine the constant A in equation (3.5), MALKUS (1956) finds that this equation takes the form

$$\frac{U_m - U}{\tau_0^{1/2}} = \frac{1}{2} g_r^2 \log (\sin \Phi) \tag{3.8}$$

where $\frac{1}{2} g_r^2$ is a constant (≈ 3) arising in the theory. This distribution is of the « velocity defect » form known to describe the mean flow in pipes and channels over a wide range of Reynolds number, it reduces to the universal logarithmic distribution for $z \ll D$, the theoretical value of the constant $\frac{1}{2} g_r^2$ is within 20 % of the experimental value, $K^{-1} = 2.5$, and the complete form agrees very well with the measurements of LAUFER (1951) in a two-dimensional channel. The velocity defect form and the logarithmic distribution near the walls are also predicted by similarity and mixing-length theories of turbulence.

The analysis of heat-transfer between parallel, horizontal planes by natural convection is very similar, but now any part of the flow in which the temperature increases upwards is definitely stable so that it is reasonable to replace the condition (iii) for Poiseuille flow by the condition

$$\frac{dT}{dz} \leqslant 0 \tag{3.9}$$

Then, expressing the mean temperature gradient in the form,

$$-\frac{dT}{dz} = \sum_{0}^{n_0} a_r \cos 2r\Phi \tag{3.10}$$

since the constant flux of total heat is

$$H = -\rho c_p \kappa \frac{dT}{dZ} + \rho c_p \overline{\omega\theta}$$

the arguments used above show that

$$\frac{dT}{d\Phi} = \frac{T_1 - T_2}{\pi (n_0 + 1)} \frac{\sin^2 (n_0 + 1) \Phi}{\sin^2 \Phi} \tag{3.11}$$

where T_1, T_2 are the temperatures of the lower and upper boundaries*.

* Again the use of circular functions rather than orthogonal functions satisfying the boundary conditions limits the validity of this expression to parts of the flow where $n_0 \sin \Phi \gg 1$.

Ignoring the oscillations as before,

$$T - T_a = \frac{T_1 - T_2}{2\pi(n_0 + 1)} \cot \frac{\pi z}{D} \tag{3.12}$$

for $n_0 \sin \Phi \gg 1$. The value of n_0 is to be determined by the condition of marginal stability and MALKUS (1960) shows that the mode of order n_0 is marginally stable in this temperature distribution at a Rayleigh number of

$$R = \frac{g}{T_a} \frac{(T_1 - T_2) D^3}{\nu \kappa} = R_c (n_0 + 1)^3 \tag{3.13}$$

where R_c is a constant depending on the nature of the boundaries. Approximate theoretical considerations indicate that its value for rigid boundaries is 2533.

From equations (3.12) and (3.13), the distribution of mean temperature outside the conduction layers is found to be

$$\frac{T - T_a}{T_1 - T_2} = \frac{1}{2\pi} \left(\frac{R_c}{R}\right)^{1/3} \cot \frac{\pi z}{D} \tag{3.14}$$

Unlike the velocity distribution (3.8), this temperature distribution has not the form predicted by similarity theories, which is

$$\frac{T - T_a}{T_a} = Q^{2/3} (g D)^{-1/3} f\left(\frac{z}{D}\right) \tag{3.15}$$

and, for $z \ll D$,

$$\frac{T - T_a}{T_a} = c' Q^{2/3} (gz)^{-1/3}$$

where $Q = \frac{H}{\rho c_p T}$ is a convenient quantity for describing heat transfer. From equation (3.11), the heat-transfer coefficient is

$$\frac{Q D T_a}{\kappa (T_1 - T_2)} = (n_0 + 1) = \left(\frac{R}{R_c}\right)^{1/3} \tag{3.16}$$

and the temperature distribution outside the conduction layers is

$$\frac{T - T_a}{T_a} = \left(\frac{\nu}{\kappa} R_c\right)^{1/2} \frac{\theta_0 z_0}{2\pi D} \cot \frac{\pi z}{D} \tag{3.17}$$

or, for $z \ll D$,

$$\frac{T - T_a}{T_a} = \left(\frac{\nu}{\kappa} R_c\right)^{1/2} \frac{\theta_0 z_0}{2\pi^2} z^{-1} \tag{3.18}$$

where $\theta_0 = Q^{3/4} (\kappa g)^{-1/4}$, $z_0 = \kappa^{3/4} (g Q)^{1/4}$. Notice the different scales of temperature variation in the defect equations and in particular the different functional forms of the predicted temperature variations for small $\frac{z}{D}$. Measurements of mean temperature in this system provide a crucial test of the Malkus theory.

4. Experimental evidence

In Poiseuille flow, the ability of the Malkus theory to describe the mean flow is not markedly superior to that of the similarity theories. These theories predict that same form of defect-law and the logarithmic form near the walls, and their prediction of the *universality* of this distribution and some related ones for rigid boundaries (Townsend, 1961) may be balanced against the numerical predictions of the Malkus theory. For this reason, experimental tests of its validy require measurements of heat convection between parallel planes or in equivalent flow systems. The simplest of these are not decisive. Both similarity and Malkus theory predict that the heat-transfer coefficient is proportional to the cube-root of the Rayleigh number, but the Malkus theory gives a fair estimate of the constant of proportionality and asserts that it is independent of $\frac{\nu}{\kappa}$. This has been confirmed experimentally by measurements of heat-transfer in air (Thomas & Townsend, 1957; Townsend, 1959) and in water and acetone (Malkus, 1954*a*).

The measurements of vertical distribution of temperature have all been made in air and, with the exception of three distributions between parallel planes (Thomas & Townsend, 1957), they have all been made without a definite upper boundary on the assumption that the distributions are similar to those with the upper surface at a great height. It is convenient to write the respective predictions for $\frac{z}{D} \ll 1$ in the forms,

$$\frac{T - T_a}{T_a} = c\, \theta_0\, z_0\, z^{-1} \qquad \text{Malkus (4.1)}$$

$$\frac{T - T_a}{T_a} = c'\, \theta_0\, z_0^{1/3}\, z^{-1/3} \qquad \text{Similarity (4.2)}$$

defining $\theta_0 = Q^{2/3}\, (g\, z_0)^{-1/3}$ and regarding z_0 as an intrinsic scale of the flow, equal to $\kappa^{3/4}\, (g\, Q)^{-1/4}$ if the surface is smooth. (Note that the value of z_0 does not affect the similarity prediction.) In the laboratory, D. B. Thomas and I have made extensive measurements of mean temperature over a smooth heated plane and, outside the conduction layer, these measurements conform well with the Malkus z^{-1} variation. The similarity $z^{-1/3}$ variation could not be fitted to the measurements and the character of the temperature fluctuations was found to change with distance from the surface in a way that is completely inconsistent with the basic assumptions of similarity theory. This was most clearly shown by changes in relative duration of the alternating « active » periods of rapid and large fluctuations of temperature and « quiescent » periods of much weaker fluctuations. The basic assumption of similarity theory is that the structure of the turbulence is everywhere similar except for scale changes and so a dimensionless quantity like the relative duration should be independent of position in the flow. As can be seen from figure 1, it decreases fairly rapidly with height above the heated surface and this is strong evidence against a self-similar structure.

The existence of active and quiescent periods in a fully developed and full turbulent flow is rather surprising, particularly as there is no corresponding effect for velocity fluctuations. It is found that the mean temperature T_a as determined by extrapolation of the mean temperature distribution, but the mean temperature during the active periods is comparatively high and these periods seem to be caused by columns of hot air which lose heat as they rise through the surrouding cold, turbulent fluid. Since these columns have their origin in the conductive-viscous layer of thickness approximately $3\, z_0$, their original distribution in space and time will be determined by the thickness and scale of the motions in this layer, and, so long as the mean temperature depends on the presence of these penetrating columns, their original distribution and the quantities specifying it will be expected to have an influence on the distribution of temperature. For this reason, it seems reasonable to regard z_0 as an intrinsic scale of the convection, important at all heights because the mean temperature is determined by the occurrence of penetrating columns originating at a height where the scales of the velocity and temperature fields are of order z_0. In the particular case of convection over a smooth plane, the intrinsic scale has the value $\kappa^{3/4}(g\,Q)^{-1/4}$ given by the thickness of the conductive layer, but there are other possibilities, e. g. it might be set by the wavelength of a corrugated surface.

The notion of an intrinsic scale as the essential difference between the pictures of heat convection presented by the similarity and by the Malkus theories has an interesting application to the problem of mixed convection between parallel planes. Within a layer, $\frac{z}{D} \ll 1$ in which shear-stress and heat-flux are nearly constant, dimensional reasoning shows that the distributions of mean velocity and mean temperature are of the forms

$$\frac{dU}{dz} = \frac{\tau_0^{1/2}}{kz} f\left(\frac{z}{L}\right) \tag{4.3}$$

$$\frac{1}{T}\frac{dT}{dz} = -\frac{Q}{k\,\tau_0^{1/2}\,z}\, g\left(\frac{z}{L}\right) \tag{4.4}$$

where $L = \tau_0^{3/2}\,(kgQ)^{-1}$ is the Monin-Obukhov length, if the direct influence of viscosity and conductivity can be neglected. This is possible not so close to the surface that viscous and conductive transfer are important, provided that the presence and properties of the viscous-conductive layer have a negligible effect on the general motion. Many experimental measurements show that this is true for layers with negligible buoyancy forces, at least so far as these mean properties are concerned, and we expect that, if there is a turbulent layer with negligible buoyancy forces separating the surface from layer from the buoyant flow, viscosity and conductivity (and surface conditions generally) will have a negligible effect on the mean gradients. Since L is a measure of the height at which generation of turbulent energy by buoyancy forces first becomes comparable with generation by shear, the motion for large values of $\frac{z}{L}$ is dominated by buoyancy effects and a regime of natural convection is expected, differing from convection with zero shear-stress only by having a velocity of translation and probably a changed value of the intrinsic scale. Since the change from dominant shear to dominant buoyancy takes place at heights of order L, the characteristic penetrating columns of natural convection arise at a height where the turbulent scale is of order L, and we would expect the intrinsic scale to be a constant multiple of L.

These conditions of constant shear-stress and heat-flux are satisfied in the lowest few hundred metres of the earth's boundary-layer over a homogeneous surface, and the temperature distributions have been studied intensively (see, for example, PRIESTLEY, 1959). In unstable conditions with $\frac{z}{L}$ more than 0.03, most of the measurements seem to confirm the similarity prediction that the gradient of potential temperature is

$$\frac{1}{T}\frac{dT}{dz} = -H_0^{3/2}\ Q^{2/3}\ g^{-1/3}\ z^{-4/3} \tag{4.5}$$

where H_0 is a constant near 1.0, but it is strange that a theoretical prediction assuming natural convection should be valid when energy production by shear greatly exceeds the production by buyoancy forces. The suspicion that the turbulent transfer in this range of $\frac{z}{L}$ is not completely independent of the wind-shear is confirmed by the work of WEBB (1958) who made measurements in conditions of strong instability and found an upper limit to the validity of the similarity prediction. Beyond this limit (approximately at $\frac{z}{L} = 0.45$), WEBB observed a rapid reduction of temperature gradient below the values predicted by the similarity theory. Figure 2 is a copy of a diagram from WEBB's paper and shows the variation of the temperature ratio

$$\frac{1}{1\cdot 5 \log 30/8}\ \frac{T_{30} - T_8}{\left(\frac{dT}{dz}\right)_{1\cdot 5}}$$

with Richardson number, $R_i = \frac{g}{T}\frac{dT}{dz}\left(\frac{dU}{dz}\right)^{-2}$, at a height of 1.5 m. (the subscripts refer to the height of measurement in metres). Lines have been drawn to show the variation of this ratio for forced convection $\left(\frac{dT}{dz} \propto z^{-1}\right)$, for similarity convection $\left(\frac{dT}{dz} \propto z^{-4/3}\right)$ and the calculated variation assuming that (4.5) is valid for $\frac{z}{L} < 0{,}45$ while

$$\frac{1}{T}\frac{dT}{dz} = -0.59\, g^{-1/3}\, Q^{2/3}\, L^{2/3}\, z^{-2} \tag{4.6}$$

for $\frac{z}{L} > 0.45$. This is exactly the Malkus prediction adapted for the mixed-convection flow by the use of an intrinsic scale, $z_0 = 0.10\,L$. The good agreement between the observations and this calculated variation shows that the mechanism of natural convection is much the same in the atmosphere and in the laboratory (TOWNSEND, 1961).

Experimentally, the Malkus predictions for heat convection are qualitatively in agreement with measurements of mean temperature in the laboratory and in the atmosphere, which is not true of the similarity predictions. For Poiseuille flow, the two sets of predictions are qualitatively similar and they are both confirmed by measurements of mean velocity. A distinctive feature of the Malkus theory is the numerical predictions of velocity and temperature distributions, not made by similarity theories, and these are in very fair agreement with experiment when it is remembered that there are no disposable constants in the theory. In the restricted class of turbulent flow between parallel planes, there seems no doubt that the Malkus theory offers a better and more consistent description than the more conventional theories, and it is an impressive achievement to have predicted a failure of the similarity theory in natural convection. On the other hand, the similarity theories offer a simple and apparently plausible explanation of the universal validity of the relation,

$$\frac{dU}{dz} = \frac{\tau^{1/2}}{kz} \tag{4.7}$$

in flows close to a solid boundary with a wide range of surface conditions (roughness, flow injection, etc.) and this is not easily shown from the basic assumptions of the Malkus theory.

5. Relation to statistical concepts of turbulent motion

So far the theory has been treated mostly as a machine for the prediction of distributions of mean velocity and temperature, without any serious attempt to relate the actual turbulent motion to the part of the motion considered by the theory. This cannot be done exactly, not because of a lack of experimental knowledge of the flows but because the theory does not specify the transporting motions with sufficient precision. Each component of the « transport » spectrum could be produced from two-dimensional motions periodic in the xOy plane and with the proper number of nodes in the vertical direction, and any combination of these is possible. Without knowing the distribution of these equivalent motions, not even the simplest parameters of the turbulent motion can be calculated without additional assumptions, and any comparison must be between qualitative features of the theoretical and observed motion.

In wall flows, many measurements, particularly those of GRANT (1958) and of FAVRE, GAVIGLIO & DUMAS (1957), show that correlation between velocities at different points may be appreciable with separations comparable with the width of the flow, even when one or both points are very close to the wall where the intense dissipation seems to demand very small sizes of eddy. The only exceptions are for the vertical component of velocity, particularly with separation in the Oy direction. Connection between motions in widely separated parts of the flow appears to be more easily explained by component motions each affecting all parts of the flow at once than by the model used in mixing-length and similarity theory which assumes eddy-sizes moderately small compared with the flow-width. On the other hand, the smaller extent of correlations of the vertical component (which is most closely associated with turbulent transport) suggests that most of the horizontal motion near the wall is a swirling motion caused by eddies whose contribution to transport is appreciable only much further from the wall, and that the remaining transporting motion could be described by the similarity theory. If this view is correct, most of the objections to using the basic result of the similarity theory for shear flow are met and one can use it successfully to describe many kinds of wall turbulence with negligible buoyance forces.

In convective flows, measurements of fluctuations are scarce though visual observations indicate penetrative movements with a range not found in shear flow, and this is clearer evidence of *transporting* movements linking separated parts of the flow than is available in wall flow. These flows are peculiar in that the original laminar instabilities are extremely persistent and tend to break up, as the Rayleigh number increases, not into ordinary turbulent motion but into cellular disturbances of higher order. This tendency probably is connected with the simpler mechanics of energy-generation from the heat flow and there is no doubt that the little we know about the fluctuations is not consistent with ideas of similarity.

The non-linear transfer of energy between components of the turbulent motion appears in a rather unusual form in the Malkus theory, but this is a consequence of using a spectral decomposition with respect to the vertical direction, equivalent in some ways to an average over the flow. In Poiseuille flow, this means that the spectrum of transporting motions covers much the same range of wave-numbers with much the same intensity as the spectrum of the nearly isotropic, dissipating eddies which possibly are in a condition of local similarity. In fact, it is well-known that most of the production of turbulent energy and most of the dissipation takes place near the edge of the viscous layer where the scale of motion is smaller than anywhere else in the flow and where the small Reynolds number of flow prevents a wide spread of eddy sizes. Further from the wall, a proper dissipation chain of locally isotropic eddies does exist but the smallest eddies of this chain are larger than the eddies at the edge of the viscous layer. With this in mind, we have the transporting modes deriving energy from the mean flow distribution as determined by their own amplitudes but not inter-acting in any other way (compare the treatment by STUART (1960) of the stability of finite disturbances of a laminar flow), and losing energy to a background of locally isotropic turbulence. For mechanical consistency, it would seem that this loss must be equivalent to an eddy viscosity of value dependent on the order of the mode concerned.

6. Developing turbulence

Turbulent flows that are not stationary in time set a difficult problem for any theory and progress has only been possible if the flow is capable of self-preserving development. In all such flows, self-preserving development is a stable, asymptotic state and initial deviations from it diminish during development as a moving equilibrium is set up. Although this equilibrium may resemble the complete equilibrium of stationary flows, the basic assumptions of the Malkus theory need considerable modification if they are to apply to self-preserving flow.

To begin with, nearly all self-preserving flows have at least one point of inflexion in their velocity profiles and the presence of a point of inflexion has very little effect on the properties of a flow. For example, a boundary-layer in zero pressure gradient has no point of inflexion but it can be described using the same flow constant (defined by equation 6.1) as boundary-layers in adverse pressure gradients which have one and sometimes two points of inflexion. This difficulty could be overcome by allowing a small number of these points, say not more than two, and it is perhaps not serious. It is interesting that one stationary flow must have a point of inflexion in its profile, plane Couette flow.

Second, the requirement of maximum energy dissipation would imply infinite rate of spread of a developing flow without some additional constraint. There is some evidence that a constraint of this kind does exist in wakes and that it affects the rate of development of instabilities of the current velocity profile. GRANT (1958) observed in a wake that sections of Kármán street developed with a spacing appropriate to the current flow width and, after entraining some ambient fluid, they slowed down and disappeared, to be replaced later by another set of Kármán eddies with wider spacing. This observation emphasies a further difficulty of basing a theory of developing flow on stability theory, that self-preserving, *time-dependent* disturbances are not compatible with the equations of motion and that continuous growth and decay of eddy structures is an essential characteristic of these flows.

The remaining two assumptions, representation of the transport by a finite series and marginal stability of the motion described by the highest term of this series, are hardly definitive in free turbulent flows which are described quite accurately for all Reynolds numbers by the first few terms of a series of the proper orthogonal functions. So it appears that developing flows are not described by the theory in its present form and that major modifications would be necessary before this could be done. In spite of this, the notion that neutral stability of a disturbance of the laminar flow has a bearing on the turbulent flow receives some support from the interesting coincidence between the turbulent Reynolds number $\frac{1}{\nu_T}\int (U_1 - U)\, dz$ (where ν_T is the effective eddy viscosity for the mean flow) and the corresponding Reynolds number for laminar instability of a wake (Townsend, 1956, p. 166), and from the successful use of the condition,

$$\frac{1}{\nu_{T_0}}\int_0^{\infty} (U_1 - U)\, dz = R_s \quad \text{the flow constant} \tag{6.1}$$

to predict development of boundary-layers and diffuser flow (Townsend, 1961*a*).

7. Conclusions

Acceptance of the concepts and assumptions of the Malkus theory is very much a matter of taste until their dynamical basis can be expressed more clearly but, within the limited class of flows so far analysed, its power of prediction has been very good. There are two reasons why some doubt must be felt about the merits of the theory as a general theory for all turbulent flows, the apparent difficulty in establishing the universal logarithmic distribution in wall flow and the inapplicability to developing flows. This suggests (to me) that the representation of the transport by components covering the whole width of the flow is a good approximation in convective turbulence, a rough one in wall turbulence and is not applicable to free turbulence.

Dr. Malkus has been kind enough to send me some details of the present state of his theory and they have been very useful, but he bears no direct responsibility for the opinions of the author.

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ON THE MALKUS THEORY OF TURBULENCE

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SOMMAIRE

Le but de la théorie de Malkus est d'éviter des calculs explicites non linéaires en faisant appel à un principe d'optimisation qui caractérise la turbulence statistiquement stationnaire. Le principe de la dissipation maximum est présenté comme celui qui peut être correct, et on montre qu'il est équivalent au transport maximum de chaleur pour la turbulence thermique. Cette discussion souligne le procédé utilisé par Malkus pour trouver l'état de transport maximum de chaleur dans un fluide turbulent transporteur thermiquement.

Le point particulier discuté est celui de la convection naturelle entre deux plaques parallèles, rigides, lisses, parfaitement conductrices.

Les champs fluctuants de vitesse et de température sont décomposés suivant le jeu de mode normal des équations linéarisées. Dans les termes de cette représentation, on discute l'aspect énergétique de la turbulence thermique, qualitativement, les effets de la non-linéarité sont dégagés.

Avec cette discussion, comme arrière plan, les hypothèses de Malkus sont introduites. Ce sont :

1) le gradient de température moyenne n'est nulle part positif,
2) une gamme finie de nombres d'ondes est efficace dans le transport de chaleur, et
3) le nombre d'ondes vertical le plus élevé contribuant au transport de chaleur est marginalement stable sur le profil de température moyenne.

Des arguments physiques sont présentés pour justifier chacune de ces hypothèses, mais la meilleure justification réside dans la présentation des résultats.

La théorie de Malkus prédétermine la forme du gradient de température observé juste à l'intérieur de la couche limite, ainsi que la forme correcte de la loi de transport de température.

De nombreuses questions concernant la théorie peuvent être soulevées, et seulement quelques-unes d'entre elles sont énumérées dans la conclusion de cet exposé.

Néanmoins, l'impression finale est que la théorie a beaucoup d'arguments favorables, et qu'elle soulève nombre de questions intéressantes concernant les écoulements turbulents.

SUMMARY

The aim of the Malkus theory is to avoid explicit nonlinear calculations by appeal to an optimization principle which characterizes statistically steady turbulence. The principle of maximum dissipation is offered as the possibly correct one and is shown to be equivalent to maximum heat transport for thermal turbulence. The present discussion outlines Malkus' procedure for finding the state of maximum heat transport in a thermally convecting, turbulent fluid.

The particular situation discussed is that of natural convection between two parallel, rigid, slippery, perfectly conducting plates. The fluctuating velocity and temperature fields are decomposed into the set of normal modes of the linearized equations. In terms of this representation the energetics of thermal turbulence is discussed qualitatively and the effects of nonlinearity are pointed out.

With this discussion as background, Malkus' assumptions are introduced. These are :

1) The mean temperature gradient is nowhere positive,
2) A finite range of wavenumbers is effective in transporting heat, and
3) The highest vertical wavenumber contributing to the heat transport is marginally stable on the mean temperature profile.

Physical arguments are offered to justify each of these assumptions, but the best justification is in the presentation of results. The Malkus theory predicts the observed form of the temperature gradient just inside the boundary layer and the correct form of the heat transport law.

Numerous questions concerning the theory may be raised and a few are listed in the concluding section of this paper. Nevertheless, the final impression is that the theory has much to recommend it and that it raises a number of interesting questions about turbulent flows.

1. Introduction

Consider a field of steady turbulence which has been decomposed into a set convenient normal modes. For each mode the ensemble mean amplitude will have a constant value resulting from a balance of three processes :

1) input;
2) molecular (and radiative) dissipation;
3) nonlinear coupling with other modes.

Energy input into a mode may result directly from external driving forces or from nonlinear coupling with a mean field of velocity or temperature. In any case the input process is a complicating factor which may depend on the special configuration of the problem at hand. For this reason much effort has gone into the study of isotropic turbulence, where the input problem is unimportant. However, even the avoidance of the input problem has not led to completely successful solution of any problem because of the well known closure difficulty from process 3.

The Malkus theory of turbulence therefore takes an entirely different viewpoint from the prevalent one of modern turbulence theory. It stresses the importance of input processes, and especially the coupling between mean fields and fluctuating modes of turbulence. Moreover, in its assumptions, it implies that the nonlinear couplings among fluctuating modes play a secondary role in the dynamics of turbulence. And finally, and most important, it seeks to avoid entirely any explicit calculation involving the nonlinear couplings.

Malkus' basic assumption is that statistically steady turbulence is characterized by the optimization of some integral property of the field of flow. This kind of assumption is not new to statistical mechanics, and even some early studies of turbulence have a similar point of departure. However, no such previous study seems to have achieved verifiable predictions, and even in theoretical explorations of the optimization principle, previous investigations have not been nearly as extensive as that of Malkus.

The theory has been attempted with two separate optimization principles. First, Malkus made the assumption that the total viscous dissipation in steady turbulence is a maximum subject to certain constraints which we shall discuss later. The other principle tried was the « relative stability criterion » of MALKUS and VERONIS (1958) which was employed only in the context of thermal turbulence. MALKUS (1961a) has recently attempted to show that the relative stability criterion implies maximum viscous dissipation, but this aspect of the theory is too detailed to be considered here. In any case, the difference between results of investigations based on these two different postulates is never vast (MALKUS, 1960).

The principle of maximum dissipation may be expressed in two equivalent ways which deserve mention. For thermal turbulence, with fixed boundary temperatures, the flow which maximizes dissipation will also transport maximum heat (MALKUS, 1954a). In his original treatment of the thermal turbulence problem, MALKUS (1954a) employed the maximum-heat-transport form of the principle. A second interesting form of the principle states that it is equivalent to maximum entropy generation for fixed boundary temperatures, and to minimum entropy generation for fixed heat flux (MALKUS, 1961a; VERONIS, 1961a). The latter statement resembles the principle of minimum entropy production derived by PRIGOGINE (1947), but MALKUS (1961a) has pointed out some distinctions.

In the present discussion no more will be said about the question of the validity of the maximum dissipation principle. It is our feeling that the principle is far from being rigorously established theoretically; nor does it seem likely that direct experimental verification will soon be possible. The criterion for the validity of Malkus' basic postulate and of his entire theory must therefore be based on the accuracy of predicted results. It is with the derivation of such results from the maximum dissipation principle that the present paper is chiefly concerned.

We shall discuss in section 4 the constraints imposed by Malkus in applying the maximum dissipation principle to turbulence arising from thermal convection between parallel plates. In particular, we shall follow the method of MALKUS' (1954a) original treatment of the problem, though the approach used in later studies (e. g. MALKUS, 1960) is somewhat different. The assumptions and approximations introduced in both treatments are, however, the same. The problem of thermal turbulence is preferred for this exposition since the calculations are generally simpler than the analogous ones for shear flow turbulence, as a reading of MALKUS' (1956) paper on shear flow turbulence will verify. For readers who may not be familiar with the problem of thermal turbulence, we present in sections 2 and 3 a qualitative summary of the physical background for the discussion to follow.

2. Preliminaries on thermal turbulence

To provide on introduction to our discussion of the Malkus theory we shall outline some of the physics of thermal turbulence in this section. The material given in this and the following section is intended to stress certain qualitative features of the problem which seem particularly germaine to an understanding of Malkus' ideas. We shall consider a fluid bounded by two horizontal planes at $z = 0$ and d. The bounding planes are rigid conductors maintained at constant temperatures. The separation between the two boundaries will be taken to be so small that the density of fluid may be considered constant except when fluctuations give rise to buoyancy effects. This last approximation is known as the BOUSSINESQ (1903) approximation in the literature on thermal convection (see also SPIEGEL and VERONIS, 1960; MIHALJAN, 1960).

In specifying the problem we shall need two further assumptions. The first is that time averages, ensemble averages, and averages over horizontal planes are equivalent. Thus the horizontal average of any quantity has no time dependence. For example, consider the total temperature T. We may write this as

$$\mathrm{T}(x, y, z; t) = \overline{\mathrm{T}}(z) + \theta(x, y, z; t) \tag{1}$$

where θ is the fluctuating component of the temperature, and $\overline{\theta} = 0$. Averages indicated by a bar will be computed as horizontal averages.

Next we must specify the nature of the boundaries. The experimental setup we have described demands that

$$w = \theta = 0 \text{ at } z = o, d \tag{2}$$

where w is the vertical component of the velocity. We shall further assume that the boundaries are stress-free. This assumption together with the continuity equation (for an incompressible fluid) implies

$$\frac{\partial^2 w}{\partial z^2} = 0 \text{ at } z = o, d. \tag{3}$$

Condition (3) is known as the free-boundary condition in the literature of thermal convection. It is probably not as realistic a condition as the vanishing of horizontal velocity on the boundary, but it greatly simplifies the calculations. (The condition of vanishing horizontal motion on the boundary is known as the rigid-boundary condition.)

The first important theoretical result about thermal convection was found by RAYLEIGH (1916) who showed that the experimental situation we have described is unstable to small perturbations whenever (see also PELLEW and SOUTHWELL, 1940)

$$\mathrm{R} = \frac{g\alpha d^3 \Delta\mathrm{T}}{\kappa\nu} \geqslant \mathrm{R}c = \begin{cases} \frac{27}{4}\pi^4 = 657 \text{ (free boundaries)} \\ 1708 \text{ (rigid boundaries)} \end{cases} \tag{4}$$

The Rayleigh number, R, is a measure of the ratio of buoyancy to viscous forces (e. g. SPIEGEL, 1960). Here α is the thermal expansion coefficient, ΔT is the temperature excess of the lower boundary over the upper, κ is the thermometric conductivity, and ν is the kinematic viscosity. Experimentally, one of the simplest properties of a convecting fluid to determine is the total heat transport, $\mathcal{H}$, through the layer. It has been found that $\mathcal{H}$ is linear in ΔT for $\mathrm{R} < \mathrm{R}_c$, but for $\mathrm{R} > \mathrm{R}_c$, $\mathcal{H}$ rises with a higher power ΔT (JACOB, 1949; MALKUS, 1954b). For very large R, the heat transport follows the law

$$\frac{\mathcal{H}}{\rho\,\mathrm{C}_p} = \mathrm{H} = -\frac{\kappa\Delta\mathrm{T}}{d}\left(\frac{\mathrm{R}}{\mathrm{R}_0}\right)^{1/3} \tag{5}$$

where R_0 is ~ 2000 (for rigid boundaries). This asymptotic law implies that H is independent of d and hence suggests the existence of a boundary layer. It is, in fact, possible to derive the form of equation (5) from boundary layer theory.

3. The dynamics of thermal turbulence

The equations governing the physical circumstances described in section 2 are the Navier-Stokes equations (including gravitational acceleration), the continuity equation, and the heat equation. By taking horizontal averages of these equations we may learn something of the mean properties of the convecting fluid. In particular we obtain from the heat equation that the heat transfer is constant throughout the fluid and is given by

$$H = \kappa\beta + \overline{w\theta} \tag{6}$$

where

$$\beta = -\frac{d\overline{T}}{dz}. \tag{7}$$

The term $\overline{w\theta}$ represents the heat flux due to convective motion.

From equation (6) we may infer the existence of a thermal boundary layer in a fluid with active convection. Since according to equation (3), $\overline{w\theta}$ must vanish on the boundary, β will have its largest value there. In the midregions, $\overline{w\theta}$ will be large, for large R, and β must be small. The effect of heat convection is therefore to distort the linear temperature profile of pure conduction to the sigmoidal profile shown in fig. 1*b*.

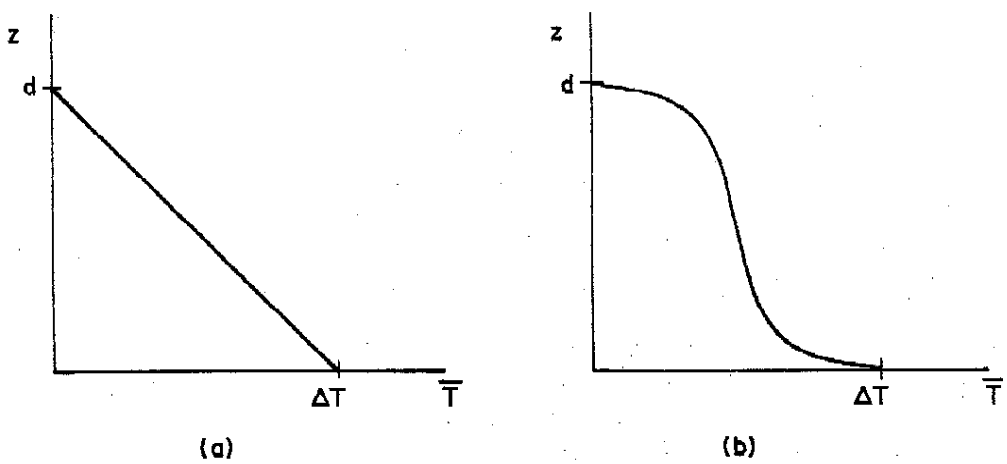

FIGURE 1
a) Temperature profile in the conductive state (no motion).
b) A representative mean-temperature profile in the turbulent state.

If we subtract the horizontally-averaged equations from the original ones we obtain a set of equations for the fluctuating quantities **u** and θ. (There is no mean velocity in this simple example of thermal turbulence.) The equations for the fluctuating quantities are (e. g. see MALKUS and VERONIS, 1958).

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\nabla^2\mathbf{u} - \mathbf{g}\alpha\theta + \frac{1}{\rho}\nabla p = -\mathbf{u}\cdot\nabla\mathbf{u} \tag{8}$$

$$\frac{\partial \theta}{\partial t} - \kappa\nabla^2\theta + \beta w = -\mathbf{u}\cdot\nabla\theta + \overline{\mathbf{u}\cdot\nabla\theta} \tag{9}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{10}$$

where p is the deviation of the pressure from its horizontally-averaged value.

On the basis of these equations we shall discuss qualitatively the energetics of thermal turbulence. In this we shall follow closely a recent approach suggested by Ledoux, Schwarzschild and Spiegel (1961). We begin by introducing a convenient set of functions in which to represent the fluctuating quantities. These functions are the normal modes of the following system of equations :

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \nabla^2 \mathbf{u} - g\alpha\theta + \frac{1}{\rho} \nabla p = 0 \tag{11}$$

$$\frac{\partial \theta}{\partial t} - \kappa \nabla^2 \theta + \beta w = 0 \tag{12}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{13}$$

Equations (11), (12), and (13) are just the equations of the problem without the fluctuation-interactions. We presume for the moment that $\beta(z)$ is a known function; but it actually is given by equation (6) and must be determined together with final solution of the problem. Equation (13) is therefore still nonlinear, otherwise equations (11) - (13) would just be the equations for an infinitesimal motion.

Equations (11) - (13) admit separable solutions. The time dependence of such solutions will be exponential and the horizontal dependence may be taken to be harmonic. The separable solutions then have the form

$$| n\,\mathbf{a}\,\eta > = e^{\eta t + i\mathbf{a}\cdot\mathbf{x}}\, \Omega_{n\mathbf{a}\eta}(z) \tag{14}$$

The eigenfunction $\Omega_{n\mathbf{a}\eta}$ represents a four-vector with three velocity components and one temperature component. The wave-vector **a** is a horizontal wavevector so that

$$\mathbf{a}\cdot\mathbf{x} = k_x x + k_y y. \tag{15}$$

The quantity n is the vertical « quantum number » which appears in the z-equation for $\Omega_{n\mathbf{a}\eta}$. The quantity n takes only integral values since the system is of finite vertical extent. The equation for $\Omega_{n\mathbf{a}\eta}$ is readily derived from equations (11) - (13) and will not be needed here (but see Appendix B).

Equations (11) - (13) describe the time behavior of the fluctuating fields under the influence of buoyant input of energy and the dissipative processes of conduction and viscosity. In the absence of the fluctuation - interactions a given mode would continue to vary exponentially in time. The growth rate, η, must be real for a convectively unstable fluid, as we shall prove in Appendix A. For each pair ($n\mathbf{a}$) there are two allowed values of η which we shall call η_+ and η_-. These two values of η correspond to physically distinct kinds of modes.

In figure 2 we show the general dependence of η_+ on **a** and n for fixed R. Where η_+ is positive, the buoyant input of energy exceeds the dissipation. For large $|\mathbf{a}|$ or n the dissipation is too large and η_+ is negative. On the other hand η_- is negative definite for all **a** and n.

The difference between the η_+ -modes and the η_- -modes is due to what we shall call the phase relations of the modes. An in-phase mode has amplitudes such that upward velocity always occurs with positive temperature fluctuation. For an out-of-phase mode upward velocity is always paired with negative temperature fluctuation. Hence buoyancy forces always act to decelerate an out-of-phase mode. Moreover, out-of-phase modes, when they are excited, convect heat downward, against the prevailing temperature gradient. In this sense, the η_+ -nodes are in phase and the η_- -modes are out of phase.

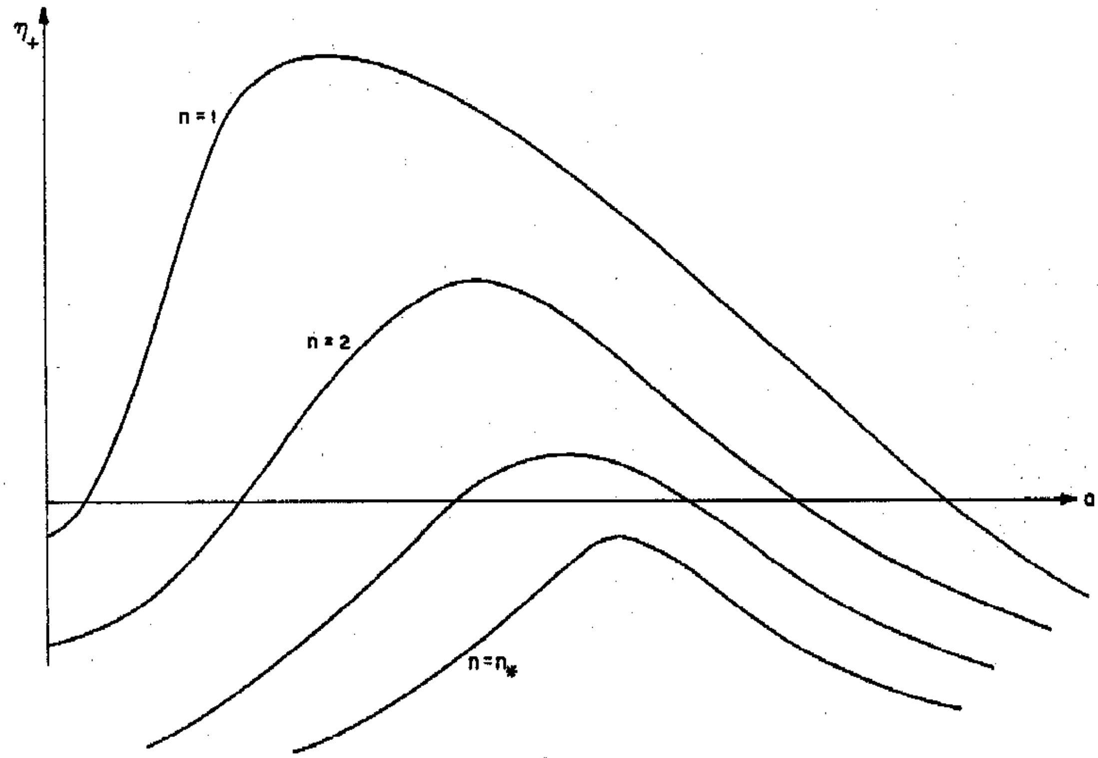

FIGURE 2
Showing η_+ as a function of n and $a = |\mathbf{a}|$.

It is convenient to introduce the notation

$$\phi_\pm = |\, n\mathbf{a}\eta_\pm >$$

We can then think of the eigenfunction $|\, n\mathbf{a} >$ as a two component function,

$$|n\mathbf{a} > = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}. \tag{17}$$

This notation is intended to make plausible the supposition, used in the remainder of this section, that the functions $|\, n\mathbf{a} >$ form a complete set into which we may expand the turbulent velocity-temperature field.

The field in question will be in a statistically steady state. When we speak of the amplitude reached by given mode, we shall always have in mind the spectral amplitude which has been ensemble-averaged.

Let us the consider the energetics of thermal turbulence in terms of the in- and out-of-phase modes. As implied in the introduction we must balance the buoyant input against molecular and nonlinear dampings. The buoyant input and molecular dissipation rates are summarized by the η's. Thus modes with $\eta_+ > 0$ are excited by buoyancy forces and drain potential energy from the system. Through nonlinear interaction, according to the bilinear terms in equation (8) and (9), η_--modes and other η_+-modes are then excited. The η_--modes are damped because they are opposed by buoyancy forces and return potential energy to the mean field. Those η_+ -modes with $\eta_+ < 0$ are also damped, but they are damped by molecular processes. The situation is summed up in fig. 3

where the curved lines indicate nonlinear interactions described by the bilinear terms in the full equations. Borrowing from the terminology of stability theory we refer in figure 3 to η_+ -modes with $\eta_+ > 0$ as unstable and with $\eta_+ < 0$ as stable.

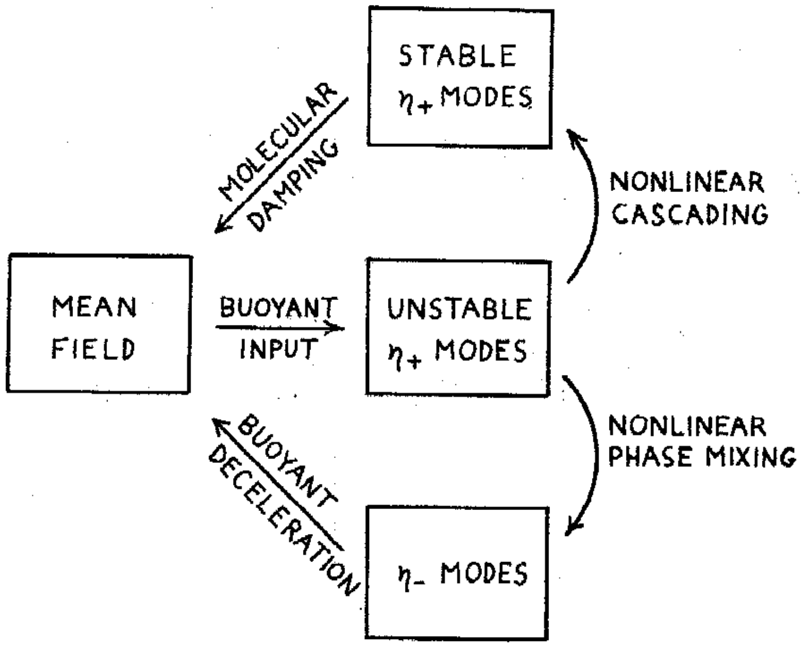

FIGURE 3
The gross energetics of convective turbulence. Arrows indicate net flows.

On the basis of this qualitative discussion of the energy balance we can sketch the power spectrum of the turbulent velocity-temperature field. In doing this we may refer to figure 2, showing the dependence of η_+ on n. We notice that there is a value n_* such that $\eta \leqslant 0$ for all $n \geqslant n_*$. Hence only η_+-modes with $n < n_*$ are directly excited by buoyancy forces and these will have largest amplitude. However, η_--modes will also have nonzero amplitude for $n < n_*$ because of nonlinear interactions. For $n \geqslant n_*$, η_+ and η_--modes will have rapidly diminishing amplitudes for increasing n, but there will still be finite excitation. Since temperature and velocity fluctuations at high wavenumber are excited by separate interactions (i. e. by $\mathbf{u} \cdot \nabla \mathbf{u}$ and $\mathbf{u} \cdot \nabla \theta$) they can occur with the same or the opposite sign almost equally often. In other words, little phase information will be transferred from mode to mode. This qualitative remark is strengthened by the localness of the nonlinear transfer; phase information is not likely to be transferred over any great distance in wave number space. It seems likely therefore that η_+- and η_--modes will have nearly equal amplitudes for $n > n_*$. In figure 4 we illustrate these remarks with a plot of the power spectra as a function of n.

Similar remarks may be made about the heat-transfer spectrum of steady state thermal turbulence. There is, however, the difference that the η_--modes convect heat in the downward direction and their presence therefore causes diminution in the convective heat transfer. In this way, the nonlinear interactions diminish convective heat transfer by exciting η_--modes. Hence, in order to maximize heat transfer we would have to minimize the strength of the nonlinear interactions.

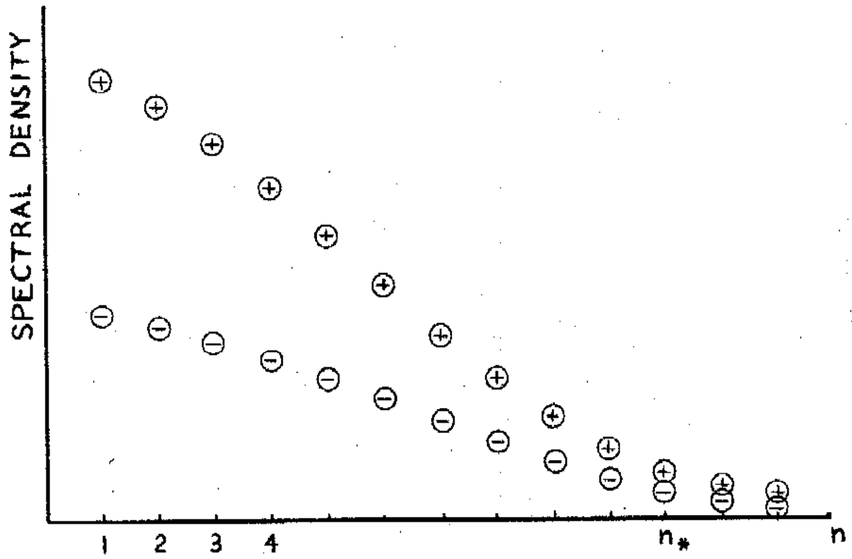

FIGURE 4
The power spectrum of the turbulent velocity-temperature field.
The curves show distribution with respect to n for fixed R and a.
The spectral densities in η_+-modes an η_--modes are plotted separately.

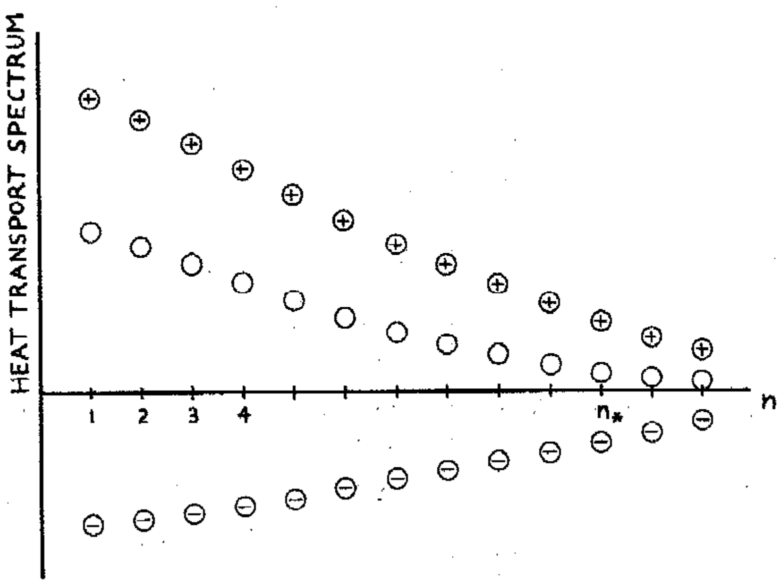

FIGURE 5
The heat transport spectrum plotted against n for fixed R and a.
The curves show energy transported by η_+- and η_--modes and their combined transport.

Figure 5 shows a plot of the convective heat transfer in each mode as a function of n. The curve of net heat transfer shown is the linear combination of the transfers in the two kinds of modes. Since for $n > n_*$ the amplitudes of η_+- and η_--modes tend to be nearly equal, the net heat transfer drops rapidly toward zero for $n > n_*$. This last property of the heat transfer is important for the ideas of MALKUS to which we now turn.

4. The Malkus theory

In this section we shall apply the MALKUS theory to the thermal turbulence problem described in section 2, following Malkus' (1954*b*) original approach to the problem. Our task then is to find that statistically steady state of turbulent motion which maximizes the heat flux, H, where H is given by

$$H = \kappa\beta + \overline{w\theta} \tag{6}$$

Some motivation for proceeding in this way was given in section I, but basically we have to take this approach as postulational.

Since the boundary temperatures are fixed, we have the additional condition

$$\int_0^d \beta \, dz = -\Delta T \, . \tag{18}$$

Another useful relation is obtained by averaging equation (6) over z:

$$H = -\frac{\kappa \Delta T}{d} + < w\theta > . \tag{19}$$

Here the symbol $< >$ denotes an average over all space as in

$$< w\theta > = \frac{1}{d} \int_0^d \overline{w\theta} \, dz \, . \tag{20}$$

Equation (19) shows that maximum H is equivalent to maximum $< w\theta >$, since the boundary conditions fix ΔT. MALKUS is guided by this fact in choosing constraints which ensure that the maximum sought for H is finite.

To understand MALKUS' choice of constraints we note that $< w\theta >$ will be finite if $\overline{w\theta}$ has an upper bound. Very likely, nonlinear interactions limit the amplitudes of w and θ and set an upper bound on $w\theta$, but we are not able to calculate such limiting amplitudes. On the other and, for finite H, $\overline{w\theta}$ will have an upper bound if β has a lower bound. MALKUS therefore makes

Assumption 1 : $\beta \geqslant 0$.

Originally, MALKUS justified this assumption on the thermodynamic grounds that heat should not flow against the prevailing temperature gradient. This argument is quite plausible but there is a difficulty in it. If we allow that any real fluid is slightly compressible we find that in the correct BOUSSINESQ form of the equations of motion (JEFFREYS, 1930; see also SPIEGEL and VERONIS, 1960) β should be

$$\beta = -\frac{dT}{dz} + \frac{g}{c_p} \tag{21}$$

where $\frac{g}{c_p}$ is the adiabatic temperature gradient. Hence, β is not strictly a temperature gradient in a real fluid.

A later justification offered by MALKUS for Assumption 1 was that in a region where $\beta < 0$ a fluctuating mode of turbulence would locally lose energy to the mean field. Hence, MALKUS would argue, the criterion for inviscid instability, $\beta \geqslant 0$, should be maintained throughout the fluid. It is, in fact, possible to give an explicit local criterion for the loss of energy of a turbulent mode to the mean field. A mode of (nondimensional) horizontal wavenumber a $(= |\mathbf{a}|)$ will lose energy to the mean field in a region where

$$\beta < \frac{a^4}{R} \frac{\Delta T}{d}. \tag{22}$$

Hence, if no mode is to lose energy to the mean field in any region of the fluid, the lower bound on β should be $\frac{\Delta T}{Rd} a_0^4$, where a_0 is the largest horizontal wavenumber contributing to heat transport. In fact, the existence of a largest horizontal wavenumber is implied by the MALKUS theory (see Appendix B). It is therefore not clear why zero should be the lower bound on β.

The next step in MALKUS' procedure of maximizing H is to expand w and θ in Fourier sine series in z. Sine series are chosen because of the boundary conditions (2) and (3). It is then possible to express $\overline{w\theta}$ in terms of these expansions. If β is symmetric about the plane $z = \frac{1}{2} d$, ordinary manipulations lead to the expression

$$\overline{w\theta} = \sum_{n=1}^{\infty} \alpha_n \sin^2 \frac{n\pi z}{d}. \tag{23}$$

The quantities α_n define the heat transport spectrum in the representation by sine functions. The maximum value of H, or $< w\theta >$, is to be sought among the great variety of possible spectra. On applying the definition (20) we find that

$$< w\theta > = \frac{1}{2} \sum_{n=1}^{\infty} \alpha_n . \tag{24}$$

For $< w\theta >$ to be finite the series of equation (24) must converge, hence the α_n must diminish rapidly for large n.

In section 3 we suggested that the heat transport spectrum falls quite rapidly for $n > n_*$ (see figure 5). In fact, if it were not for the nonlinear interactions, the spectral amplitudes would be zero for $n \geqslant n_*$. We also argued that nonlinear interactions tend to diminish heat transport by exciting the out-of-phase modes. We concluded that to maximize heat transport we should minimize the strength of nonlinear interactions; this was even proven by MALKUS and VERONIS (1958) for small Rayleigh number. A consequence of minimizing the strength of the non linear interaction would then be the minimization of the amplitudes in the tail of the heat transport spectrum.

The remarks of the preceding paragraph are appropriate only when the representation in terms of the modes of section 3 is made. They are offered here to make plausible MALKUS'

Assumption 2 : There exists an n_0 such that $\alpha_n = 0$ for all $n > n_0$.

In any case, Assumption 2 ensures the convergence of series (24) and hence the existence of a finite maximum for H. However, it need not imply cutoffs in the velocity and temperature spectra; KOLMOGOROFF's laws should still be satisfied at large wavenumbers.

With Assumptions 1 and 2 as constraints, H may be maximized by varying the α_n. In his original treatment, MALKUS (1954*a*) introduced Assumption 1 by assuming that β actually goes to zero somewhere in the fluid. The heat transport spectrum then turned out to be

$$\alpha_n = -4\,\kappa \frac{\Delta T}{d}\left(1 - \frac{n}{n_0 + 1}\right), \qquad n \leq n_0 \tag{25}$$

while the heat transport is given by

$$H = \kappa \frac{\Delta T}{d}(n_0 + 1)\,. \tag{26}$$

These results are not meaningful without the specification of n_0, but a comparison with experiment is possible at this stage through the corresponding form of β :

$$\beta = -\frac{\Delta T}{d}\,\frac{1}{n_0+1}\,\frac{\sin^2 (n_0+1)\dfrac{\pi z}{d}}{\sin^2 \dfrac{\pi z}{d}}\,. \tag{27}$$

TOWNSEND (1959) has measured the temperature distribution in a convecting fluid, just outside of what he calls the conducting layer. Equation (27) enables us to predict the distribution in that layer. If we assume $n_0 \gg 1$ and take $\frac{z}{d} \ll 1$, we find that, away from the boundaries,

$$T(z) = T\left(\frac{1}{2}\,d\right) + \frac{d}{2\pi(n_0+1)}\,z^{-1}\,. \tag{28}$$

This dependence on z^{-1} is especially interesting since PRIESTLEY (1954) has obtained a $z^{-1/3}$ dependence by a similarity argument. TOWNSEND's data decide in favor of the z^{-1}-law of the MALKUS theory. It is interesting that this particular success of the theory seems to depend very sensitively on Assumption 2.

In order to fix H we must prescribe n_0. The same reasons which led to Assumption 2 provide a basis for selecting n_0. The existence of modes for $n \geq n_*$ was due, we saw, to the nonlinear interaction. In maximizing heat transport we were led to minimize the effect of these higher modes on heat transport. If this reason for truncating the heat transport spectrum is sound, we should naturally adopt

Assumption 3 : $\qquad n_0 + 1 = n_*$.

Or, to put it in words, the $(n_0+1)^{th}$ mode is to be the mode closest to neutral stability on the mean temperature field.

Unfortunately, there is an inconsistency implied in Assumption 3 which can best be understood by considering equations (11) - (13). These may be reduced to a single equation for w (see Appendix A). The condition for neutral stability is obtained by setting $\frac{\partial}{\partial t} = 0$ in this equation. We then have

$$\nabla^6 w = \frac{g\alpha}{\kappa\nu}\,\beta\,\nabla_1^2 w \tag{29}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{30}$$

Equation (29) in non dimensional form is

$$\nabla^6 w = \mathrm{R} \left| \frac{\beta}{\frac{\Delta \mathrm{T}}{d}} \right| \nabla_1^2 w \tag{31}$$

where d is taken as the unit of length. Equation (31) has solutions of the form

$$w = e^{i\mathbf{a}\cdot\mathbf{x}}\, \mathrm{W}_{na}(z) \tag{32}$$

where **a** is the horizontal wavevector, as before. The function W_{na} satisfies the equation

$$\left(\frac{d^2}{dz^2} - a^2\right)^3 \mathrm{W}_{na} = -\mathrm{R}_{na} \left| \frac{\beta}{\frac{\Delta \mathrm{T}}{d}} \right| a^2\, \mathrm{W}_{na}$$

in which R_{na} is the eigenvalue of R. The boundary conditions are discussed in section 2.

Consider now a particular experimental situation with a specified value of the Rayleigh number. The smallest value of n for which $\mathrm{R}_{na} \geqslant \mathrm{R}$ for all **a** is called n_*. By Assumption 3, n_* is identified with $n_0 + 1$. The inconsistency arises becauses MALKUS makes his expansions in terms of sin $n\pi z$, which in general is different from W_{na} both in z-dependence and in being independent of **a**. Assumption 3, however, implies that the n's are equivalent, in some sense, for both functions. Moreover, the physical arguments we have put forward to justify Assumptions 2 and 3 require for their validity that expansions be made in terms W_{n*}, or related functions.

It seems possible to restate the theory in terms of appropriate expansions (see Appendix B). In this way, Malkus' approach may be considered an approximation to the more carefully posed problem. To clarify this suggestion let us note that the WKBJ approximation for W_{na} gives

$$\mathrm{W}_{na} = \mathrm{N}_{na}\, h^{-1/2} (h^2 + 1)^{-1/2} \sin \left[\frac{n\pi \int_0^z h\, dz}{\int_0^d h\, dz} \right] \tag{34}$$

where N_{na} is a normalization constant and

$$h = \left[\left(\frac{\mathrm{R}\beta}{a^4 \frac{\Delta \mathrm{T}}{d}} \right)^{1/3} - 1 \right]^{1/2} \tag{35}$$

Expression (34) will not be accurate unless

$$\beta \gg \frac{\Delta \mathrm{T}}{d} \frac{a^4}{\mathrm{R}} \tag{36}$$

which is valid in the boundary regions. Also, β is nearly constant near the boundaries and hence, sin $n\pi z$ should be a good approximation to a W_{na} there. Since Malkus' calculation depends almost entirely on properties of the boundary region, no effect of the inconsistency of Assumption 3 should appear in his results. However, any use of his formulation to obtain information about the midregion is likely to be less accurate.

To obtain n_0 Malkus used a variational principle. The main contribution to the integral comes from the boundary region and his use of sine functions is adequate for the reasons just given. His calculation yields (MALKUS, 1961b)

$$R = R_0 (n_0 + 1)^3 \tag{37}$$

where

$$R_0 = \begin{cases} 1533, \text{ free boundaries} \\ 2533, \text{ rigid boundaries} \end{cases} \tag{38}$$

We then obtain

$$H = \kappa \frac{\Delta T}{d} \left(\frac{R}{R_0} \right)^{1/3} \tag{39}$$

as would be expected. A discussion of the numerical agreement of R_0 and the constants in equation (28) is given by TOWNSEND (1959); the slope predicted by the theory is reasonably accurate, but the additive constant in equation (28) disagrees by more than the experimental error would allow.

5. Concluding remarks

One objection which may be raised to the Malkus theory is that the optimum state which he has found may not correspond to any solution of the equations of motion. In recent work, MALKUS (1960) has attempted to add constraints which ensure that the optimum state be allowed by the equations of motion. In particular, he requires that the velocity and temperature field satisfy the so-called power integrals, or second moments, of the equations. An even more recent study by L. N. HOWARD (unpublished) seems to indicate that the use of the power integrals as constraints implies a cutoff in the heat transport spectrum. This is not surprising since the power integrals do not contain any influence of the fluctuation-interactions. One suspects that the introduction of further constraints based on higher moments of the equations will permit the heat transport spectrum to extend weakly to infinity. Though Malkus' original optimum state is probably not allowed by the equations, it is perhaps a limit of a sequence of allowed states. The current work should clarify this possibility.

A second difficulty may also arise in cases where the fluctuation-interactions are essential to the dynamics of the motion. For example, in the limit of very small Prandtl number the equations of thermal turbulence reduce almost exactly to the equations of homogeneous turbulence, except for the presence of an input term (SPIEGEL, 1962). In particular if the Prandtl number σ satisfies

$$\sigma R \ll R_c \tag{40}$$

the convective heat transport will be negligible and the mean temperature field will not be distorted by the motion (LEDOUX, SCHWARZSCHILD, and SPIEGEL, 1961). The difficulty is that there is no indication in the Malkus theory of the possible change of the character of the flow as the Prandtl number decreases and the relative importance of nonlinearity increases. Of course, there may be no real discrepancy here, but certainly $< |\mathbf{u}|^2 >$ derived with the Malkus theory differs markedly in Rayleigh number dependence from the value obtained using the idea of eddy viscosity. However, the calculation of $< |\mathbf{u}|^2 >$ in both approaches has been approximate and the objection raised may be premature. More careful study of this limiting case is called for.

There is also a question of generalizability of the theory to complicated situations occurring in nature. One perplexing type of configuration met in astrophysics and geophysics entails a β which is strongly negative in the absence of motion. Convection can diminish the region of negative β, but not remove it entirely. How, in such instances, are we to apply Assumption 1 of the Malkus theory ?

We should also call attention to MALKUS' (1956) study of shear flow using the ideas presented here. The shear flow study is much more difficult than that of thermal turbulence especially because the determination of n_0 brings the Orr-Sommerfeld equation into the problem. Nevertheless, Malkus has succeeded in predicting the correct velocity-defect law and a reasonably accurate value of the von Karman constant. The theory is limited, however, in shear flow as well as in thermal turbulence in the quantities which may be predicted. Information on velocity spectra, for example, does not yet seem obtainable from the Malkus theory.

In spite of these questions which remain, we should like to conclude this discussion with the judgment that the Malkus theory has provided a powerful new approach to the problem of turbulence. It is rich in physical insight and may even shed light on the problems of irreversible processes in general. Certainly, it calls for intensive further study, especially along experimental lines. Experiments in thermal turbulence are particularly needed; the possibility of checking Malkus' assumptions with accurate spectral data promises to be very interesting.

I shall not attempt to list all of the people with whom I have enjoyed valuable discussions of the problems touched on in this report, but I should like to acknowledge my indebtedness to these discussions.

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APPENDIX A

We wish to prove here that Im $\{\eta\} = 0$ when Re $\{\eta\} \geqq 0$, where η is the growth-rate appearing first in equation (14). To do this simply let us reduce equations (11), (12) and (13) to a single equation for w.

Taking $\nabla \times \nabla \times$ of equation (11) we obtain

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) \mathbf{u} - g\alpha \nabla^2 \theta + g\alpha \nabla \frac{\partial \theta}{\partial z} = 0 \tag{A1}$$

where we have used equation (13). If we then take $\left(\frac{\partial}{\partial t} - \kappa \nabla^2\right)$ of the z-component of equation (A1) and add $g\alpha \nabla_1^2$ of equation (13) to the result we find

$$\left(\frac{\partial}{\partial t} - \kappa \nabla^2\right)\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) \nabla^2 w = g\alpha\beta \nabla_1^2 w \,. \tag{A2}$$

Equation (A2) admits separable solutions of the form

$$w\,(x, y, z\,; t) = e^{\eta t + i\mathbf{a}\cdot\mathbf{x}}\, W_{n a \eta} \,. \tag{A3}$$

The equation for $W_{na\eta}$, without subscripts, is

$$\left[\eta - \kappa\left(\frac{d^2}{dz^2} - a^2\right)\right]\left[\eta - \nu\left(\frac{d^2}{dz^2} - a^2\right)\right]\left(\frac{d^2}{dz^2} - a^2\right) W = - g\alpha\beta a^2 W \,. \tag{A4}$$

On multiplying equation (A4) by W^* and integrating over z we find

$$\eta^2 I_1 + \eta I_2 + I_3 - I_4 = 0 \tag{A5}$$

where

$$I_1 = \int_0^d \left[\left|\frac{dW}{dz}\right|^2 + a^2 |W|^2\right] dz \tag{A6}$$

$$I_2 = (\kappa + \nu) \int_0^d \left|\frac{d^2 W}{dz^2} - a^2 W\right| dz \tag{A7}$$

$$I_3 = \kappa\nu \int_0^d \left[\left|\frac{d}{dz}\left(\frac{d^2 W}{dz^2} - a^2 W\right)\right|^2 + a^2 \left|\frac{d^2 W}{dz^2} - a^2 W\right|^2\right] dz \tag{A8}$$

$$I_4 = g\alpha a^2 \int_0^d \beta\, |W|^2\, dz \,. \tag{A9}$$

Suppose now that

$$\eta = p + iq \tag{A10}$$

where p and q are real. The imaginary part of equation (A5) gives

$$q\,(2pI_1 + I_2) = 0. \tag{A11}$$

Hence, if p is positive $q = 0$. Of course, η has two roots and for one of them p is negative definite. However, since the roots must be either real of complex conjugates, $q = 0$ for both when one p is positive.

The two roots of equation (A5) are

$$\eta_{\pm} = \frac{I_2}{|2I_1|}\left[\pm\sqrt{1 - \frac{4I_1(I_3 - I_4)}{I_2^2}} - 1\right].$$

APPENDIX B

To remove the inconsistency from Assumption 3 let us reexamine the expansion procedure which led to equation (23). We begin by expanding w and θ in Fourier series in the horizontal direction :

$$w(x, y, z, t) = \sum_{\mathbf{a}} \theta_{\mathbf{a}}(z, t)\, e^{i\mathbf{a}\cdot\mathbf{x}} \tag{B1}$$

$$\theta(x, y, z, t) = \sum_{\mathbf{a}} w_{\mathbf{a}}(z, t)\, e^{i\mathbf{a}\cdot\mathbf{x}} \tag{B2}$$

These should strictly be integrals in equations (B1) and (B2), but we have in mind box-normalization.

Suppose now we introduce complete sets of functions consistent with conditions (2) and (3). We can then write

$$w_{\mathbf{a}}(z, t) = \sum_{n=1}^{\infty} B_{n\mathbf{a}}(t)\, \Psi_{n\mathbf{a}}(z) \tag{B3}$$

$$\theta_{\mathbf{a}}(z, t) = \sum_{n=1}^{\infty} A_{n\mathbf{a}}(t)\, \Phi_{n\mathbf{a}}(z) \tag{B4}$$

when $\Phi_{n\mathbf{a}}$ and $\Psi_{n\mathbf{a}}$ are members of the complete sets. If we multiply w by θ^* and integrate in the horizontal we now obtain

$$\overline{w\theta} = \overline{w\theta^*} = \sum_{a}\sum_{nn'} H^{\mathbf{a}}_{nn'}\, \Phi_{n\mathbf{a}}\, \Psi_{n\mathbf{a}} \tag{B5}$$

since the integration in the horizontal produces the function $\delta(\mathbf{a} - \mathbf{a}')$.

Here the matrix

$$H^{\mathbf{a}}_{nn'} = A_{n\mathbf{a}}\, B_{n'\mathbf{a}} \tag{B6}$$

has no time dependence since we have earlier assumed that horizontal averaging removes time dependence.

If $H^{\mathbf{a}}_{nn'}$ were a finite matrix we could put it into canonical from with terms only on the main diagonal avel the diagonal just above this. This cannot in general be done for an infinite matrix, but we offer conjecture here that it is possible to put $H^{\mathbf{a}}_{nn'}$ into canonical form by a similarity transformation. Correspondingly, Φ_{na} and Ψ_{na} will be transformed to new functions, W_{na} and Θ_{na}. For $\overline{w\theta}$ we then have

$$\overline{w\theta} = \sum_{a} \sum_{n} [\alpha_{na} W_{na} \Theta_{na} + \gamma_{na} W_{na} \Theta_{n+1,a}] \tag{B7}$$

In order now to give Assumption 3 precise meaning we take W_{na} and Θ_{na} to be eigenfunctions of the neutral stability equations for vertical velocity and temperature. These equations are

$$\left(\frac{\frac{\Delta T}{d}}{\beta}\right)\left(\frac{d^2}{dz^2} - a^2\right)^3 W_{na} = - R_{na}\, a^2 W_{na} \tag{33}$$

and

$$\left(\frac{d^2}{dz^2} - a^2\right)\left(\frac{\frac{\Delta T}{d}}{\beta}\right)\left(\frac{d^2}{dz^2} - a^2\right)\Theta_{na} = - R_{na}\, a^2 \Theta_{na} \tag{B8}$$

One property of these eigenfunctions is that they are alternately odd and even for successive n. Since β must be even, we can conclude that $\gamma_{na} = 0$ and

$$\overline{w\theta} = \sum_{a} \sum_{n=1}^{\infty} \alpha_{na} W_{na} \Theta_{na}\,. \tag{B9}$$

Equation (B9) is intended to be a generalization of Malkus' form for $\overline{w\theta}$. Correspondingly we may generalize Assumptions 2 and 3. First we have

Assumption 2′ : For every **a** there exists an $n_0(\mathbf{a})$ such that $\alpha_{na} = 0$ for $n > n_0(\mathbf{a})$

Then,

$$\overline{w\theta} = \sum_{a} \sum_{n=1}^{n_0(a)} \alpha_{na} W_{na} \Theta_{na}\,. \tag{B10}$$

Assumption 3 then becomes,

Assumption 3′ : For each **a** let $n_*(\mathbf{a})$ be the smallest value of n such that $R_{na} \geqslant R$ where R is the (given) Rayleigh number of the problem. Then $n_0(\mathbf{a}) + 1 = n_*(\mathbf{a})$.

Assumptions 2′ and 3′ imply that there are finite ranges of n and **a** contributing to the heat transport. In figure 6 we indicate the region in wavenumber space which contributes to the heat transport spectrum. The sketch shows the $\frac{k_x d}{\pi} - n$ plane and the lines indicate the values of $\frac{k_x d}{\pi}$ and n which we sum over in equation (B10).

To complete the theory we should maximize H with respect to the α_{na} subject to Assumption 1. The procedure is difficult to carry out, however, and so far has been possible only in the approximate form used by Malkus. This form is obtained by letting W_{na} and Θ_{na} be $\sin \frac{n\pi z}{d}$. In this case **a** disappears from view and the maximization becomes feasible.

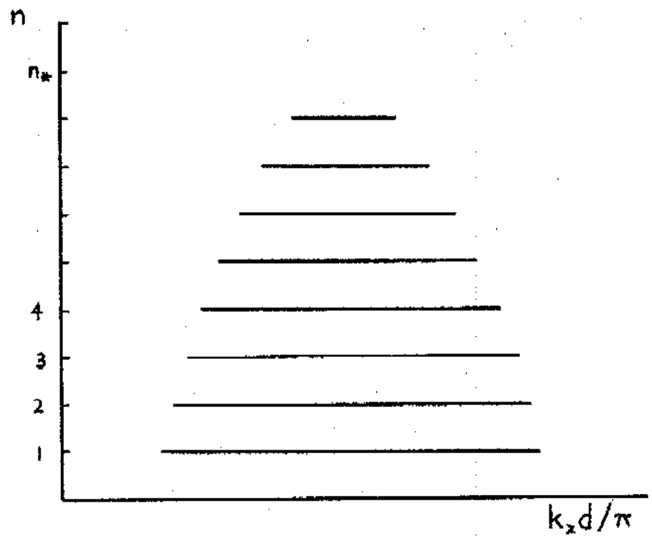

FIGURE 6
Indicating the cells in k space which contribute to the heat transport in the generalized statement.
Only a projection on the $n - \dfrac{k_x d}{\pi}$ plane is shown;
the complete region is a figure of rotation symmetric about the n-axis.

INTERVENTIONS

KOVASZNAY. — How is this applied to shear flow ?

SPIEGEL. — In one of the papers cited (MALKUS, 1956) the problem of Poiseuille flow in a channel is treated. A second paper on the subject* shows the strong parallel between the channel flow and the thermal problems. The quantity corresponding to β is $-\dfrac{d^2 U}{dz^2}$ where U is the fully-developed, mean velocity profile and z is the coordinate transverse to the channel. In analogy with Assumption 1, $-\dfrac{d^2 U}{dz^2}$ is required to be positive-definite.

Malkus has employed a somewhat different expansion procedure in the channel problem than we have described, but the general approach and the assumptions are retained. One must find n_0 in exact analogy to the thermal turbulence problem; this requires the solution of the Orr-Sommerfeld equation with the fully developed profile. Solutions of this kind for higher modes in the z-direction have not been extensively studied**, and there are some difficulties in calculating n_0. However, Malkus has devised an approximation technique to find n_0 and has succeeded in deriving the familiar logarithmic velocity-defect law and an acceptable prediction of von Karman's constant.

KESTIN. — How is the theory related to the stability diagram ?

SPIEGEL. — Presumably for each higher mode there is a neutral stability loop in the $R_e - \alpha$ plane where R_e is the Reynolds number of the mean flow and α is the " downstream " wavenumber. I have never seen such loops calculated for n greater than 1, but if the analogy

* W.V.R. MALKUS, Transactions of the Brussels Meeting of the IUTAM. 1956.
** But Professors KESTIN and MORKOVIN have kindly pointed out the work of D. GROHNE, NACA TM 1417, Dec. 1957.

to thermal turbulence holds, they should look like the sketch shown in fig. a. The dashed vertical line shows the value of Reynolds number in the experiment.

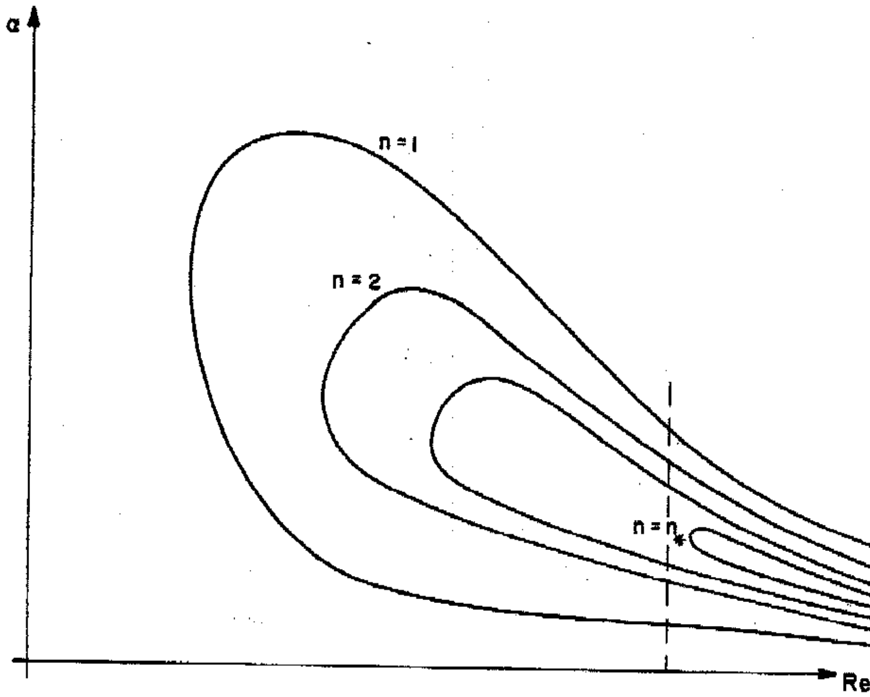

FIGURE *a*
Curves of neutral stability for several "transverse" modes in the shear flow problem. The dashed vertical line indicates the given Reynolds number of the problem.

In the thermal problem the stability diagram is as shown in the second sketch, fig. b. This second sketch is related to the discussion of section 3. If we imagine that the coordinate perpendicular to the R — a plane is η_+, the surfaces η_+ (R, a) intersect the plane in the neutral stability curves. Figure 2 is then a trace of $\eta_+ - a$ plane at the value of R indicated by the horizontal line in fig. b.

LIEPMANN. — With no inflections (i.e. with $-\dfrac{d^2 U}{dz^2} \geq 0$) there is no instability. What do these stability considerations then mean ? Betchof : (In answer to Liepmann).

Perhaps, though the laminar profile is not unstable to infinitesimal disturbances, the fully developed profile is.

SPIEGEL. — Dr. Betchof's conjecture is the one on which the theory must rest. Presumably, the initial instability of the laminar profile is possible only through a finite amplitude disturbance, but infinitesimal disturbances may well be able to draw energy from the fully developed mean profile.

BATCHELOR. — The cutoff in the heat transport spectrum should not be at n_*. An interval beyond n_* should be allowed for information to be lost.

SPIEGEL. — This question of how local in wavenumber space are the nonlinear interactions perhaps cannot be answered precisely. But there are two points that I should like to bring up in response to Dr. Batchelor's remark.

The first is that the kind of information which must be transferred is what we have called the phase relations between w and θ. Since the higher modes of w and θ are generated by separate nonlinear terms (i.e. by $u \cdot \nabla u$ and $u \cdot \nabla \theta$), I would think that the phase relations are lost after a very few interactions.

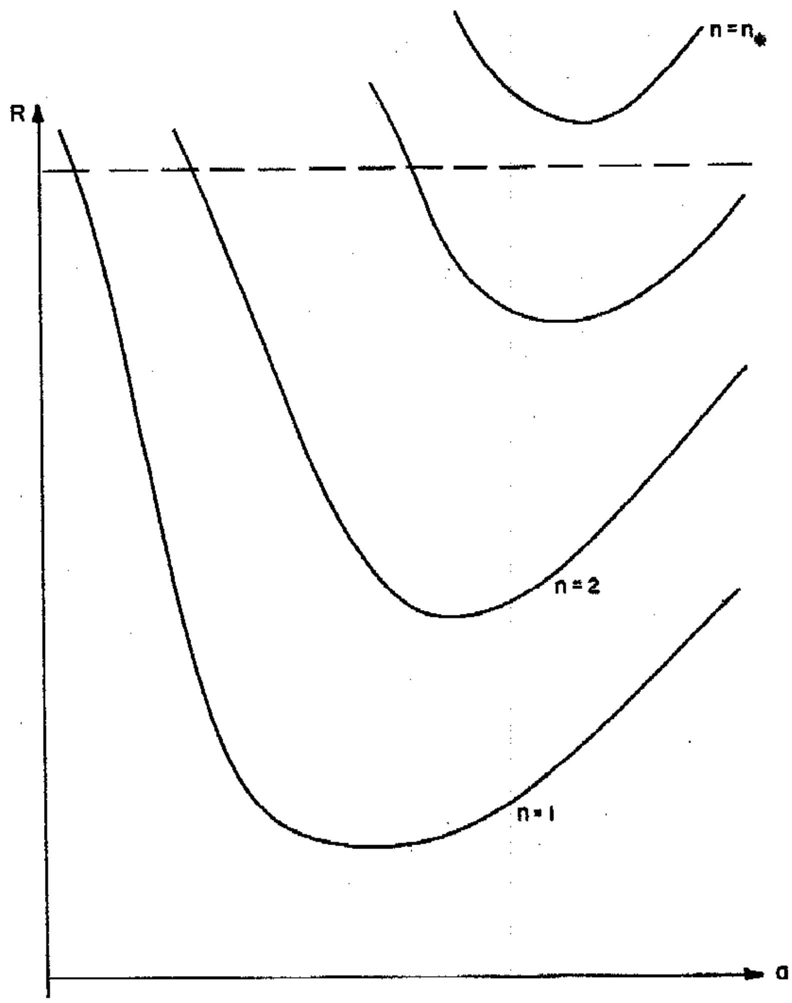

FIGURE *b*
Curves of neutral stability for several "vertical" modes of the thermal convection problem. The dashed horizontal line indicates the givenRayleigh number of the problem.

Secondly, the modes just above n_* are probably not much excited by those just below n_* since the latter themselves do not have large amplitudes. Presumably the heat transport tail derives much of its energy from modes with $n \ll n_*$. Thus even in the region just above n_* the excitation has traveled over some range in n.

In addition to these qualitative and somewhat vague justifications, there is the need to satisfy the heat transfer law :

$$H = \frac{\Delta T}{d}\left(\frac{R}{R_0}\right)^{1/3}.$$

To derive this law in the Malkus theory it is necessary that

$$(n_0 + 1)^3 \propto R.$$

Since we know that this kind of dependence results from the stability argument, we are almost forced to cut off the spectrum at n_*.

DISCUSSION DE LA SECTION : TURBULENCE STATIONNAIRE PLEINEMENT DÉVELOPPÉE

Professeur A. ROSHKO, Secrétaire scientifique

Professor Stewart opened the session and introduced Professor Hinze's paper with the following remarks :

One of the most striking features of turbulence, and yet a feature not usually made central in theoretical studies, is its " structural " stability. Thus in a long pipe or channel, or in a steady state turbulent convection system there is a remarkable contrast between the detailed behaviour and the statistical behaviour. For example in a pipe we find that no matter how precisely we specify the initial values and the boundary conditions, we are quite unable to predict the detailed motion for long times after the time of initial specification. The reason is clear — it is the fact that in strongly non-linear problems small causes do not give rise to small effects.

On the other hand, we find that no matter how crudely we initiate the flow, or how drastically we distort it, after a comparatively short time all the statistical properties of the flow closely approach their steady state values. This is the feature of turbulence which is most clearly illustrated by steady, fully developed turbulent flows.

It is this feature which W. V. R. Malkus has sought to exploit in his theoretical work, and I have therefore chosen to devote this session largely to an exposition of his work. To begin with, I have asked Professor J. O. Hinze to examine the experimental data to which any theory must conform. There have been certain common interpretations of these data which are inconsistent with Malkus's ideas, and it is particularly important to see whether the data themselves demand these interpretations, or whether alternative views are admissible.

Professor Hinze then gave his address. At the conclusion of Professor Hinze's talk Professor Stewart suggested that discussion be postponed until after the prepared papers, and called immediately for the next presentation.

The author of the second paper Dr. A. A. Townsend, was not present at the Colloquium; his paper was presented by Dr. I. C. B. Nisbet.

At the conclusion of the talk, Doctor Maurin wanted some clarification of hypothesis (2) particularly as to what happens to modes higher than n_0. Professor Stewart pointed out that all modes can exist, but only those up to n_0 contribute to $\overline{uw}$. Professor Kestin asked whether Dr. Nisbet could relate these modes to the usual stability diagram (i e. in the wave number-Reynolds number plot). Dr. Nisbet attempted to do this, there were several interjections from the floor, then Dr. Spiegel gave his interpretation.

Dr. Spiegel was then called upon to give his discussion of Malkus's ideas. Following Dr. Spiegel's talk, which had not been written out at the time of the colloquium, there was considerable discussion and questioning of several points. A clarification of these has been included in the written report subsequently prepared by Dr. Spiegel. The following is merely a summary of the points raised :

Effectivement, les équations de la convection étant linéarisées pour que l'on en sache former des solutions, la perturbation thermique, dont on étudiait la stabilité, apparaissait comme le produit d'une fonction exponentielle du temps par une fonction trigonométrique des coordonnées; de sorte que dans le plan elle s'étendait à l'infini, et que, ne constituant pas, dans le milieu, l'obstacle que l'on peut contourner, elle ne pouvait subir la poussée d'Archimède due à la différence de densité qu'entraînait sa différence de température.

L'équation de la convection, relative à la coordonnée verticale, ne peut donc contenir l'indispensable terme $g\alpha T$, origine de la convection; on se trouvait donc dans cette situation assez exceptionnelle en physique, que la forme adoptée pour la solution mettait en défaut les équations du problème. Autrement dit, le problème traité par les mathématiciens était étranger à celui qu'il aurait fallu résoudre.

Comme on ne pouvait envisager de ne pas linéariser, la situation semblait sans issue. Je me suis tiré d'affaire, en première approximation, en étudiant indépendamment l'instabilité thermique et l'instabilité mécanique; et j'ai rapproché les deux points de vue, pour obtenir le critère d'apparition de la turbulence, en jouant à la fois sur les coordonnées eulériennes et les coordonnées lagrangiennes, ce qui était légitime dans cette approximation.

Le résultat a été la mise en accord, dans des conditions inespérés; de la théorie, et des mesures de Bénard; la théorie a même montré que les tourbillons des diverses formes pouvaient avoir une infinité de dimensions (et même ne pas avoir pour section droite un polygone régulier); j'ai expliqué ainsi, avec une approximation de 1 %, les diverses dimensions observés par Bénard, et qui étaient restées mystérieuses.

Il est vraisemblable que diverses théories linéaires sont passibles de l'objection ici présentée.

COMMENTAIRE DE LA SECTION :
TURBULENCE STATIONNAIRE PLEINEMENT DÉVELOPPÉE

Professeur R. W. STEWART
Président

The session on steady, fully developped turbulence was devoted to a discussion of the work of W.V.R. MALKUS, whose theoretical studies on turbulence are peculiarly suited to such turbulent fields. Three formal papers were presented : an appraisal of the experimental results by J.O. HINZE, and discussions of MALKUS's concepts with respect to shear flow and convective turbulence by A. A. TOWNSEND (presented by I. C. B. NISBET) and by E. SPIEGEL.

MALKUS starts from four assertions or principles none of which have as yet been derived from other well established physical principles.

These assertions are :

1. Mean fields of velocity and temperature can approach, but never exceed, the condition for marginal stability of a fluid with no viscosity. (It should be noted that in the case of shear flow the condition is approached from the stable side — i.e. the curvature of the flow does not change sign; while in convective turbulence it is approached from the unstable side and the temperature gradient does not change sign).
2. There exists a smallest scale of motion which contributes to the transport of momentum or of heat.
3. There exists an integral condition on the flow. For example this may be a maximization, consistent with the other assertions, of the rate of energy release for a fixed volume of pipe flow.
4. The smallest scale of assertion 2 is that scale of motion which is marginally stable on the mean field of the fully developed turbulent flow.

These 4 principles provide a complete theory from which the flow may be predicted without adjustable constants. In the case of a shear flow assertion 4 requires the use of the Orr-Sommerfeld equation and thus severe mathematical difficulties arise. In the case of convective turbulence the mathematics is much more straight-forward, but the experiments for comparison are not so good as might be hoped. Nevertheless the results of such comparisons as can be made show remarkably close agreement, not only qualitatively but quantitatively.

It should be noted that assumption 4 implies that the molecular constants ν and K are of importance throughout the flow even at high Reynolds numbers. This conflicts with the usual assumption of Reynolds number similarity and the concept of " rough flow ". Hinze's examination of the experimental pipe flow data showed that although these assumptions are not bad, the data still allows the interpretation that Reynolds number dependence never disappears.

MALKUS's theory of convective turbulence predicts a temperature profile which varies as Z^{-1} near the wall (but away from the linear region right against the wall). The result, which depends upon assertion 2 only, conflicts with the usual similarity result $T \propto Z^{-1/3}$.

The experimental observations still leave much to be desired, but there seems to be some support for MALKUS's result. His theory of the shear flow yields a logarithmic profile for the mean velocity, as does the similarity argument. The similarity arguments in the two cases can be made so parallel (see Appendix) that if one is in question it casts great doubt on the validity of the other.

There was considerable discussion during the session of the validity and importance of the cut-off of assertion 2.

It turns out that even a small " tail " after the cutoff importantly influences the results, and the agreement with experiments seems to depend critically on an end to the transport spectrum which is sharp indeed.

Townsend showed, in his paper, that a simplified approach using assertions 1 and 2, (althought the choice of orthogonal functions, for the expansion which assertion 2 cuts off, was arbitrarily taken to the trigometric — a choice which would be hard to defend logically) together with a simple assumption which closely approaches assertion 3, yields profiles qualitatively very close to observed ones. Since he did not employ assertion 4 or any equivalent he could not make quantitative predictions.

APPENDIX

Similarity arguments showing the close analogy of the shear flow and pure convection cases :

	Shear Flow	*Convection*
1.	Stress $\overline{uw} = u_*^2$ is constant.	Heat flow $\overline{w\theta} = H$ is constant.

2. The only scale length is the height Z above the surface.

3. Rate production of turbulent energy at any level :

	Shear Flow	*Convection*
3.	$= u_*^2 \dfrac{dU}{dZ}$	$\propto g\,H$
4.		$\theta' \propto Z \dfrac{dT}{dZ}, \therefore w' \propto \left(Z \dfrac{dT}{dZ}\right)^{-1}$

5. Rate of dissipation of turbulent energy :

	Shear Flow	*Convection*
5.	$\propto u_*^3 Z^{-1}$	$\propto w'^3 Z^{-1}$
6.	$\therefore u_* \dfrac{dU}{dZ} \propto \dfrac{u_*^3}{Z}$	$\therefore w' Z \dfrac{dT}{dZ} \propto \dfrac{w'^3}{Z}$ or $\left(\dfrac{dT}{dz}\right)^3 \propto Z^{-4}$
7.	$U \propto \ln Z$	$T \propto Z^{-1/3}$

* The "prime" symbol is used to signify root mean square.