SECTION

TRANSFERT D'ÉNERGIE EN TURBULENCE HOMOGÈNE

ENERGY TRANSFER IN HOMOGENEOUS TURBULENCE

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THE DYNAMICS OF HOMOGENEOUS TURBULENCE: INTRODUCTORY REMARKS

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Cette vue d'ensemble, brève et superficielle, des travaux concernant la dynamique de la turbulence homogène est destiné à donner une idée de l'état actuel du problème. Le travail est examiné sous trois titres suivants : les tourbillons porteurs d'énergie, les procédés suggérés pour les calculs concernant la turbulence homogène, et les propriétés de la turbulence à petite échelle.

SUMMARY

This brief and superficial review of work on the dynamics of homogeneous turbulence is intended to convey an impression of present understanding of the problem. Work is reviewed under the following three headings : the energy-containing eddies, suggested procedures for the calculation of homogeneous turbulence, and the small-scale properties of the turbulence.

The notion of transfer of energy between components of turbulent motion characterized by different length scales is a very old one. The interaction between different components caused by the non-linearity of the equation of motion cannot be described completely by the transfer of energy, but energy transfer is physically its most important manifestation. We may take the scope of this session to include all aspects of the dynamical interaction between different components, with particular reference to the transfer of energy.

When the mean velocity of the fluid is steady but not uniform, the mean and the fluctuating velocities may both be regarded as composed of a set of components characterized by different length scales, and transfer of energy may occur between components of the mean velocity field and components of the fluctuating velocity field, as well as by interaction of components of the fluctuating velocity among themselves. Some recent theoretical work has been based on the supposition that the interaction of the components of the fluctuating velocity field is here comparatively unimportant, at any rate so far as the components with large length-scales are concerned. Whether this supposition be valid or not, it remains desirable that we should understand the process of self-modulation of the fluctuating velocity field. This is the only kind of interaction to occur in homogeneous turbulence, and we may expect it to play an important part either when the turbulent motion is not affected appreciably by the presence of rigid boundaries, as in many geophysical and astrophysical contexts, or when the length scales of the interacting elements are small, in any turbulent flow.

For many years after 1935, most of the fundamental work on turbulence was concerned with homogeneous turbulence, this kind of turbulence being both convenient for experimental study and apparently simpler theoretically in view of the absence of the mean velocity field. This relative simplicity is not so apparent nowadays, and there are welcome signs that the concentartion on homogeneous turbulence is being replaced by an attack on a wider front. Considerations of the wider strategy of turbulence research will no doubt arise later in the colloquium; so far as this session is concerned, the objective is a critical essessment of the progress that has been made towards an understanding of the dynamics of homogeneous turbulence. This first paper in the session is intended only as a broad introduction to the session, and has no pretensions to being a comprehensive survey.

A noticeable feature of work on homogeneous turbulence during the last 20 years is the increasing degree of subdivision and refinement of the problem. We can now recognize many different aspects and characteristic properties of homogeneous turbulence, many of which call for their own appropriate analysis and theories. A field of homogeneous turbulence does not present a single problem which we might hope to describe by means of a single theory or mathematical solution. It presents a many-sided face, and has many recognizable typical features, just as laminar flow fields do, and inevitably the trend in research is toward isolation and separate treatment of these many features. The division of the whole range of eddy sizes into large, or energy-containing, eddies on the one hand and smaller eddies on the other is now a familiar idea; there is also a distinct range of sizes in which most of the mean-square vorticity resides, and a range such that the remaining contribution to the velocity field is effectively a uniform straining motion. Each of these ranges has acquired its own set of observations, concepts, hypotheses and results. We have also learnt to recognize and analyse features related to the continual extension of material lines, and features related to the smaller relaxation times of eddies of smaller size. This process of 'divide and conquer' may seem to be a poor alternative to the production of a grand all-embracing theory of turbulence, but it is probably the appropriate course of development. At all events, theories with limited objective ad scope are all we have at the moment, and all we seem likely to have, apart from very general and concise, but probably powerless, formulations of the dynamical problem of the kind developed by Hopf (1952), so that we must make the most of them.

Before considering several of the current ideas, let us recall the bare analytical outline of the dynamical problem. It seems to be generally agreed that it is possible to define a Fourier transform $\mathbf{A}(\mathbf{n})$ of the velocity $\mathbf{u}(\mathbf{x})$ with respect to the position co-ordinate $\mathbf{x}$ (more precisely, to define the integral of the Fourier transform over a finite region of wave-number space), despite the fact that $\mathbf{u}$ is a stationary random function of $\mathbf{x}$. I think that some of the mathematical issues involved in this operation need further consideration, one or two of which are not simply a need for justification of a procedure believed heuristically to be valid; for instance, it would be useful to know how the Fourier transform is affected by the observed tendency for the energy of small-scale components to be confined to certain randomly distributed regions of the fluid. However, on taking for granted the existence of the Fourier transform of $\mathbf{u}(\mathbf{x})$, suitably generalized in the stochastic and Stieltjes senses, we may write formally

$$\mathbf{u}(\mathbf{x}) = \int \mathbf{A}(\mathbf{n}) e^{i\mathbf{n}\cdot\mathbf{x}}\, d\mathbf{n}\,,$$

and find, from the equation of motion, that

$$\frac{\partial A_j(\mathbf{n})}{\partial t} = i \int n_k A_k(\mathbf{n}-\mathbf{n}') \left\{ -A_j(\mathbf{n}') + \frac{n_j}{n^2} n_l A_l(\mathbf{n}') \right\} d\mathbf{n}' - \nu n^2 A_j(\mathbf{n})\,,$$

where the integration with respect to $\mathbf{n}'$ is over the whole of wave-number space. Expressions may now be obtained for the rate of change of the energy spectrum tensor [which is proportional to $\overline{A_i(\mathbf{n})\, A_j^*(\mathbf{n})}$] and of other averaged moments of $\mathbf{A}$. This expression for $\dfrac{\partial A_j}{\partial t}$ reveals the triad character of the inertial interaction of Fourier components and the related dependence of the energy transfer on statistical relationships between the phases of Fourier components.

The history of research on the dynamics of homogeneous turbulence is for the most part a record of guesses or hypotheses about the consequences of the non-linear term in this equation which render the mathematical problem partially tractable. After about 20 years of patient enquiry, it is relevant to ask : is it likely that the dynamical problem will ever be solved without the aid of hypotheses ? I am inclined to think that it will not, at any rate not in any comprehensive way, in view of the many effects of essentially different character represented in the equation. The prospects for a strictly deductive solution seem to be best in the case of the asymptotic state in which $(\overline{\mathbf{u}^2})^{1/2} \dfrac{L}{\nu} \to \infty$ and $nL \gg 1$ (where L is a length representing the size of the energy-containing eddies), and even here such a solution is not within sight.

It is also relevant to ask if progress on the purely observational side is so rapid that a proper mathematical solution is unlikely to be needed. Here I think the answer is definitely no, mainly because there are too many quantities needed in a reasonably complete description of the turbulence for observational coverage to be possible (especially if we take into account the smallness of the number of people willing and able to make measurements of turbulent flow). The experimental scene has indeed been made brighter very recently by success in attempts to measure temperature fluctuations in the very high Reynolds number turbulence in a tidal channel off Vancouver Island by Grant and his colleagues at the Pacific Naval Laboratory, and by a promising start in the measurement of rapid fluctuations of salt concentration in water, by people at Stanford University; measurements of the spectrum and other properties of these two quantities should supplement very usefully measurements of velocity fluctuations made with a hot-wire anemometer. However, these and other prospective developments do not alter the fact that even a partial deductive solution would be a most valuable preliminary to a thorough understanding of the problem.

The energy-containing eddies

During the 10 years beginning in 1945 there was a spate of work concerned with properties of the eddies, or Fourier components, in the range defined roughly as that containing most of the kinetic energy. Many measurements were made of the rate of decay of the total energy per unit volume in turbulence generated behind a grid of rods or bars placed in a uniform stream; the degree of spherical symmetry was observed, and measurements were made of the form of the spectrum, and of several of its integral moments, and a few were made of the so-called triple velocity correlation. Theories were devised to account for many of the significant features of these observations, some with and some without success, and a reasonably coherent body of data backed by some analysis and plausible physical argument was accumulated. A great deal was accomplished, although there can be no doubt that the problem presented by energy-containing eddies is not really solved and that it is too artificial a problem to justify a continuation of intensive effort on the same scale. Left in the hands of experimenters and people making direct use of the data, the problem might have fulfilled its original promise of providing a relatively simple case of turbulence which would allow the development of clear ideas about the effect of eddies of different sizes on each other; but it was too attractive (despite being too difficult) a problem for mathematically-minded workers, and a large number of the theoretical contributions to the problem now look both remote from reality and uninteresting.

One useful by-product of the period of concentration on homogeneous turbulence has been the firm establishment of statistical terminology, of the analytical methods and concepts of probability theory, and of Fourier analysis. There was a time when the use of such concepts was termed the " statistical theory ", as if this was one of several possible methods of approach. The current view, I think, is that the use of correlation and spectrum functions, etc., is inevitable and that random function theory provides the appropriate analytical framework for a description of turbulent motion. Research on homogeneous turbulence, and on the properties of the energy-containing eddies in particular, put random function theory on the turbulence map, and we may confidently expect that the same analytical concepts will become the accepted framework for work on non-homogeneous and shear-flow turbulence.

Much of the work on the energy-containing eddies in decaying homogeneous turbulence was based on the hope that there might exist a similarity state, in which all functions describing the energy-containing eddies would preserve their form (perhaps only approximately) during the decay although with length scales which vary as some power of t. An hypothesis that such a similarity state exists yields a prediction about the law of decay of total kinetic energy, although it leaves undetermined the self-preserving form of the various correlation functions. The experimental evidence gives partial support to the similarity hypothesis, but as more data has accumulated it has become evident that it is only partial and that the similarity hypothesis cannot be more than a rough approximation to the actual state of affairs in turbulence generated by a grid in a uniform stream. The main reason for this is not difficult to find. The characteristic time of the group of energy-containing eddies is $\frac{L}{(\overline{(u^2)}^{1/2})}$ and is the same as the time for on appreciable decay of energy, since decay takes place by the generation of small eddies resulting from the interaction of energy-containing eddies among themselves. Consequently the time which would be required for a similarity state of the energy-containing eddies to be established, after the turbulence has been created with particular initial properties dependent on the nature of the grid used, is of the same order of magnitude as the decay time, and there is insufficient time available for the erasure of the particular initial properties of the turbulence. For instance, if the turbulence happens not to be isotropic immediately downstream of the grid (as appears to be the case for grids of certain shape), it is not to be expected that it will become closely isotropic before the turbulence has decayed appreciably. This is not to say that a similarity state cannot exist; it may be capable of existing, but the initial conditions would need to be chosen to conform with this similarity state if the similarity state is to be observed in practice.

Even if it happens that a similarity state for decaying turbulence is capable of existing and that it can be established in a uniform stream by a suitable choice of the generating grid, there remains a question about the utility of the similarity state. It has to be admitted that decaying homogeneous turbulence does not occur often in nature (no doubt because of its transient existence) or in technology, quite apart from the need for a special set of initial conditions if the similarity state is to be realized. Approximately homogeneous turbulence which is maintained by a continual supply of energy occurs more often, but here the properties of the energy-containing eddies are directly determined by the nature of the mechanism supplying the energy. I think we must conclude that little more useful information can be obtained from theoretical analysis of assumed similarity states of the energy-containing eddies, and that further useful developments are likely to require methods of analysing temporal changes in the statistical properties of the energy-containing eddies — which makes the problem about as difficult as steady non-homogeneous turbulence.

It is convenient to place under this heading of the energy-containing eddies a remark about the results obtainable from expansion of the spectrum function, and of other Fourier transforms of correlation functions, in powers of the ware-number, on the assumption that all relevant integral moments of the velocity correlation functions exist. Substitution of such expansions in the dynamical equation yields the energy decay law and the form of the spectrum function at very large values of t when the intensity of the turbulence is small and inertia forces are no longer significant, and yields also certain constraints on the " big eddies " during earlier stages of the decay. Later work by PROUDMAN and REID (1954) revealed, in another context, that an assumption that all integral moments of correlation functions continue to converge is not consistent with the dynamical equation. It seems that, owing to the fact that the pressure at any point in the fluid is determined by a Poisson-type equation and is consequently influenced by the whole velocity field, a statistical connexion between the pressures at two widely separated points always develops, and that this in turn produces long-range connexions in the velocity field. Certain integral moments of the velocity correlation are then not convergent. The results which are legitimately obtainable from expansion in powers of the wave-number have been carefully re-examined (BATCHELOR and PROUDMAN, 1956), and the one aspect of homogeneous turbulence which was believed to provide quick returns from simple mathematics is now seen to be almost as horribly complex as other aspects; the only remaining definite prediction is that the total energy ultimately decays, as $t^{-5/2}$ in the final period of decay, which is close to the result found experimentally. This is probably a closed chapter in the history of work on homogeneous turbulence, and is mentioned here as an advertisement of the danger presented by the comforting but usually invalid assumption of convergent integral moments of velocity correlations.

Suggested procedures for the calculation of homogeneous turbulence

A number of analytical procedures have been suggested for the calculation of the spectrum function in homogeneous turbulence and its variation with time, all of which are capable in principle of any desired degree of accuracy. In practice, it is doubtful if any of them can give much analytical information of real interest; nor have they yet contributed significantly to our physical understanding of turbulence. For these reasons I shall list the suggestions with only brief comment.

(*a*) Direct numerical integration of the governing equations.

Rather surprisingly, the idea of simply making a numerical forward integration with respect to t for one realization of a field of homogeneous turbulence, with an arbitrary initial distribution of $\mathbf{u}$, seems not to have been explored thoroughly. Some years ago, Emmons (1947) attempted to calculate the temporal change in the velocity distribution in one realization of turbulent flow between parallel planes, although the number of steps in t was very severely restricted by the numerical techniques used. In these days of high-speed computers there is scope for much more extensive calculations. It would be worth-while, I think, to carry out a numerical integration of the vorticity equation in one realization of a field of homogeneous turbulence, and to attempt to verify directly some of the speculations which have been made about the small-scale structure of the turbulence. The numerical task is enormous, but time and the continual improvement of computing machines are on our side here.

(*b*) Expansion in powers of t.

The idea here is, in effect, to write $\mathbf{u}(\mathbf{x}, t)$ as a Taylor series in $(t - t_0)$:

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t_0) + (t - t_0)\left\{\frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t}\right\}_{t_0} + \frac{1}{2}(t - t_0)^2 \left\{\frac{\partial^2 \mathbf{u}(\mathbf{x}, t)}{\partial t^2}\right\} + \dots,$$

in which the coefficients can be expressed entirely in terms of $\mathbf{u}(\mathbf{x}, t_0)$ and its derivatives with respect to $\mathbf{x}$ by use of the equation of motion. Averages of products of the values of $\mathbf{u}$ at different values of $\mathbf{x}$, or their Fourier transforms, may now be expressed as power series in $(t - t_0)$, and can be computed, in principle, when the mean values of velocity products at time t_0 are known. The structure of the above series for $\mathbf{u}$ is such that the coefficient of $(t - t_0)^n$ involves terms of the rth degree in $\mathbf{u}$ or its spatial derivatives, where r takes all values from 1 to $n + 1$; consequently, the coefficient of $(t - t_0)^n$ in the power series for, say, the double velocity correlation $\overline{u_i(\mathbf{x}, t)\, u_j(x'\, t)}$ involves moments of the velocity at time t_0 of various orders up to $n + 2$. In other words, the number of velocity moments which must be known at time t_0 in order to make possible a calculation of a mean quantity at time t increases with $t - t_0$. We might suppose, for the purposes of the calculation, that second-order moments of $\mathbf{u}$ are " known " at time t_0, by a combination of observation and estimation, and possibly also third and fourth order moments, but about higher order moments we have virtually no information.

Consequently it is not possible to compute more than two or three terms in the Taylor series.

The method of expansion in powers of t is suitable for an examination of the immediate dynamical modification of some simple initial condition, such as that the spectrum consists solely of a single line at some wave-number magnitude. The more important task of investigating the development of some kind of asymptotic state (as $t \to \infty$) for arbitrary initial conditions is of course beyond the scope of the method.

(*c*) Expansion in powers of the Reynolds number $\mathcal{R}$.

It has occasionally been suggested that expansion of mean velocity products as a series of positive powers of either $\mathcal{R}$ or $\mathcal{R}^{-1}$ might be useful, although I know of no interesting results obtained in this way. A series in powers of $\mathcal{R}$ would be hopeless at the high Reynolds numbers normally relevant to turbulence, and its use seems to be limited to very large values of t when decaying homogeneous turbulence is on the point of extinction. A series in powers of $\mathcal{R}^{-1}$ is not likely to reduce the basic difficulty of coping with the non-linear terms in the equation of motion, because the zero-order term in such a series retains the full impact of the non-linearity and is out of reach of calculation.

(*d*) Replacement of the velocity distribution of $\mathbf{u}$ by its first n moments.

This seems to be the most promising of the various calculational procedures, and it has certainly yielded the largest number of results, although the limitations are severe. In as much as low-order moments of a probability distribution contribute more information than high-order moments, we can obtain an approximation to the joint probability distribution of the values of the velocity at two or more positions and times by determining the first few moments of the distribution alone, with an arbitrary assumption about values of the moments of higher order. In this way the number of unknown functions specifying the joint probability distribution is made finite and a closed finite set of dynamical equations can be written down. Success in practice in obtaining a good approximation to the true solution clearly depends on being able to retain as dependent variables a sufficiently large number of the low-order velocity moments (two being the largest number retained in published work) and on a helpful choice of the values of the higher-order moments which are not being retained as variables.

In this latter connexion, there is some experimental evidence (Batchelor, 1953; Uberoi, 1953) to suggest that, so far as the effect of the energy-containing eddies is concerned, the joint probability distribution of $\mathbf{u}$ at two different values of $\mathbf{x}$ is roughly normal, that is, that cumulants (normalized with an appropriate power of the mean-square velocity) beyond the second are roughly zero; a normal distribution is a plausible property of the velocity contributed by the energy-containing eddies, since they presumably still have some of the independence associated with their generation by a grid or some similar means. This suggests that a good choice of the values of the cumulants not being retained as variables is zero.

On the above basis (and with fourth and higher-order cumulants of the velocity at two different values of $\mathbf{x}$ put equal to zero), a number of calculations (Proudman and Reid, 1954; Tatsumi, 1957) have been made concerning the decay of homogeneous turbulence. These exploratory studies have not yet yielded much new information, but they have shown that the approximations employed are capable of yielding an initial distribution of the rate of energy transfer which in some respects is sensible.

The idea of calculating fourth-order product mean values of various kinds from an assumption that the corresponding cumulant is zero has been used in other connexions. When the fourth-order product mean value contains only simultaneous values of the velocity, and is used for the calculation of properties of the energy-containing eddies, either at the given initial instant or after a small time interval, the assumption seems to be reasonably safe (although more experimental evidence is needed before novel aspects of the results can be accepted). When used in other ways, it becomes a step in the dark and the results are of uncertain significance; indeed some of the results obtained by use of the assumption are definitely unplausible (CHANDRASEKHAR, 1955, 1956; KRAICHNAN, 1961; O'BRIEN and FRANCIS, 1962).

e) KRAICHNAN's analytical procedure.

A method of estimating the effect of the non-linear interaction between different Fourier coefficients has recently been suggested by KRAICHNAN (1959). It is claimed that the method is but the first step in a series of successive approximations, so that it can be grouped under the same general heading as the procedures just mentioned. Dr. PROUDMAN is making a critical assessment of the idea in this same session, and I shall say no more here.

The small-scale properties of the turbulence

We may designate the Fourier components for wave-numbers of magnitude $n \gg \frac{1}{L}$ as comprising the " small-scale " components of the motion. It is a familiar observation (TAYLOR, 1938) that, for sufficiently large Reynolds numbers, most of the mean-square vorticity is contributed by Fourier components with wave-numbers in this range, and that the centre of gravity of the contributions to mean-square vorticity moves towards larger wave-number magnitudes, as the Reynolds number increases, without change in the form of the spectrum in the energy-containing range. In other words, the centre of dissipation is far from the centre of energy, and the decay process involves transfer of energy, by inertial interaction, over a large range of wave-numbers. These facts led KOLMOGOROFF (1941) to formulate the hypothesis that the small-scale components of the motion are in a universal equilibrium or similarity state determined by the viscosity ν and the rate of energy dissipation per unit mass ε. The theory and its immediate consequences are well-known and need not be reviewed here. Our concern is more with its validity, as assessed in the light of current ideas and the available data, and with its place in the subject today, 20 years after publication of the theory.

So far as formal deduction is concerned, the theory is as much of a hypothesis as it was at the beginning. Despite the extreme simplicity and generality of the hypothesis, no deductive analytical argument has yet shown it to be true or false, and I know of no line of argument which gives any promise of doing so. However, the rational, and physical content of the theory have been given a great deal of thought, and I think it is fair to say that it grows more and more appealing. It was an indication of the physical plausibility of the theory that VON WEIZSÄCKER (1948), and ONSAGER (1945), and I understand OBUKHOFF (1941) also, should have independently conceived the same basic idea, or one very close to it, and since then the ideas of the theory have gained wide acceptance. The universal similarity theory of the small-scale components has an intrinsic naturalness and "rightness", and stands as one of the landmarks in the development of fluid mechanics. This is not to say that I cannot conceive of the theory being wrong; I am asserting that, relative to the state of knowledge in 1941 (and also in 1961), the universal similarity theory makes possible a great jump forward in our understanding of turbulence, and that, if it should prove to be wrong in whole or in part, the reasons for this would be almost as interesting as the theory itself.

Observational tests of the theory have been sought during the last 20 years, but only recently have useful comparisons between theoretical predictions and observations become possible. Dr. ELLISON will be presenting an account of the available data concerning the spectrum and the theoretical implications, later in this session, and, as you will have seen from summary of his paper, the $n^{-5/3}$ law predicted by the universal similarity theory in the inertial sub-range of wave-numbers stands up well to the comparison. Earlier observational tests of the theory were mostly rendered inconclusive by the insufficiently large Reynolds number attained in the experiments. The theory is an asymptotic one, and its predictions hold with increasing accuracy (if the theory is correct) as $\mathcal{R} \to \infty$, but no theoretical estimate has been made of the actual value of $\mathcal{R}$ needed for a given degree of accuracy. Consequently it is necessary to obtain from the measurements themselves information about whether the high-Reynolds-number conditions assumed in the theory (e. g. extensive separation of the energy and vorticity spectra) do in fact hold. It seems, in the light of a number of experimental investigations of homogeneous turbulence generated by regular grids, that it is almost impossible, in normal wind tunnels, to obtain a sufficiently large Reynolds number for an appreciable inertial sub-range to exist; nor is this range any more likely to be established in a shear-flow turbulence under laboratory conditions. The less extreme requirement that an equilibrium or similarity range should exist can be met more readily in the laboratory, but here the theory makes less specific predictions which are not easily checked by experiment. The fact that the high Reynolds numbers to which the theory is relevant are achieved in many turbulent flows occurring naturally in the atmosphere and in the ocean has directed attention towards these flows, but the difficulties in carrying out controlled experiments are here very severe and only very recently has a significant comparison between the theory and observations in a natural turbulent flow been made (GRANT, STEWART, and MOILLIET, 1962).

The theory makes predictions about quantities other than the form of the energy spectrum, and it is desirable that some of these should be subjected to searching observational tests. Especially interesting would be measurements at very large Reynolds numbers of some of the dimensionless moments of the probability distribution of velocity derivatives, all of which should be absolute constants according to the theory. The available measurements of the quantity

$$\frac{\overline{\left(\dfrac{\partial u}{\partial x}\right)^3}}{\left\{\overline{\left(\dfrac{\partial u}{\partial x}\right)^2}\right\}^{3/2}},$$

which is relevant both to energy transfer rates and to rates of extension of vortex lines, suggest that it is still decreasing (in magnitude) at the highest Reynolds number of the measurements, and we cannot yet be sure that the asymptotic value is non-zero. Quite apart from the testing of the universal similarity theory, measurements of moments of velocity derivatives might throw some light on the structure of the small-scale motion. We know that the flatness factors

$$\frac{\overline{\left(\frac{\partial^n u}{\partial x^n}\right)^4}}{\left\{\overline{\left(\frac{\partial^n u}{\partial x^n}\right)^2}\right\}^2}$$

show a remarkable increase as n increases from 0 to 4 (Batchelor and Townsend, 1949), for which only a tentative explanation is available. This explanation supposes that the energy associated with a Fourier component for a large wave-number is confined to certain randomly distributed portions of the fluid, whose total volume is a dimishing fraction of the whole volume of the fluid as the wave-number increases. In other words, it is suggested that there is an increasing degree of spottiness in the spatial distribution of energy of small-scale components. Further experimental investigation (of the kind carried out recently by Sandborn (1959) in a turbulent boundary layer) is needed before the explanation can be accepted and before we can be said to have an understanding of the small-scale structure of turbulence.

No short assessment of the place of the universal similarity theory today is complete without a remark about the extensive use to which the theory has been put in a wide range of physical problems. A common initial reaction to the theory was that, although it was intrinsically interesting, it would not have much practical value since it said nothing about the eenrgy-containing eddies and important quantities such as rates of transport of momentum and heat. Since then we have become aware of the large number of physical processes to which the small-scale components of a turbulent motion are relevant, and the view of utility of the theory has changed. Here is a partial list of the various problems to which application of the universal similarity theory has been made:

(*a*) Relative dispersion of floating particles and marking agents in the atmosphere;
(*b*) Spectrum of electron density in the ionosphere and consequent magnitude of scattered radio waves of short wave-length;
(*c*) Break-up of drops of one liquid immersed in another in turbulent motion;
(*d*) Mixing of two fluids in turbulent motion;
(*e*) Generation of magnetic field in clouds of ionized hydrogen in the galaxy.

A pattern of procedure which has been used on several occasions in theoretical work, and promises to be used even more often in future, is to combine the universal similarity theory with an auxiliary hypothesis in order to make possible some fairly definite predictions about the small-scale properties of turbulence. Several important groups of such auxiliary hypotheses can be recognized.

Each of the auxiliary hypotheses in the first group prescribes a relation between the rate of transfer of energy across wave-number n in the equilibrium range and the energy spectrum function (of which that suggested by Von Weizsäcker (1948) and Heisenberg (1948) is perhaps the best known), and thereby allows the calculation of the form of the spectrum function over the whole of the equilibrium range. Dr. Ellison will be describing the degree to which these predicted spectrum forms are consistent with the available data. In this connexion, I think we should keep in mind that the contribution to our understanding of the non-linear transfer process that is made by a transfer hypothesis which happens to give a spectrum function close to what is observed depends on the clarity of the physical principle or mechanism underlying the hypothesis. If this physical foundation is obscure or non-existent, we can infer only that such-and-such a relation between the transfer and spectrum functions seems to represent the facts correctly; we might feel disposed to go ahead and use the same transfer function under other conditions, outside the range of available measurements, and make further predictions, but this is largely a gain of convenience and leaves the why and the how of the original success unexplained.

A second and smaller group of auxiliary hypotheses which have yielded specific results when combined with the universal similarity theory are of a Lagrangian character and concern the random straining motion experienced by a material element of linear dimensions small compared with $(\nu^3/\varepsilon)^{1/4}$, which is the representative size of the vorticity-containing eddies. It seems to be purely a matter of kinematics that such an element becomes long and narrow and tends continually to align itself with the direction of the greatest instantaneous positive rate of strain, and thereby to be extended at a rate which makes only moderate fluctuations about a non-zero positive average. By assuming, as an approximation (which may not affect the form of the results), that these fluctuations in the rate of extension are absent (as in fact was first suggested by Townsend (1951*b*) from data concerning the rate of cooling of a hot material element of fluid in turbulent flow) it is possible to determine the form of the spectrum of a convected scalar quantity of small diffusivity (compared with ν) in the equilibrium range of wave-numbers (Batchelor, 1959). Similar methods are being used in an investigation of the spectrum of a convected vector quantity with small diffusivity. It has been appreciated for some time that a Lagrangian view of non-linear interactions may be helpful in a qualitative way (for example, in the matter of production of mean-square vorticity by extension of vortex lines); the more recent work uses Lagrangian considerations quantitatively, at any rate in investigations of the interaction between the velocity field and that of some convected quantity. Townsend (1951*a*) has used such ideas in the construction of a model of the flow at length-scales small compared with $(\nu^3/\varepsilon)^{1/4}$, with a view to representing one aspect of the self-interaction of the velocity field, but the model is not free from difficulties, one of which is that continued straining of a material body of fluid in which vorticity is a stationary random function of position leads to indefinite increase of the mean-square vorticity despite the action of viscosity (Pearson, 1959).

The related problem of determining the spectrum, in the equilibrium range, of either a convected scalar or a convected vector quantity of large diffusivity has also been tackled successfully (Batchelor, Howells and Townsend, 1959; Moffatt, 1961), although by different methods, not of a Lagrangian character.

Concluding remarks

It is our business at this colloquium to make a dispassionate assessment of the present state of knowledge of turbulence, and I presume that some kind of picture will emerge as a result of our deliberations. The considered views of most people here have yet to be heard, and it is solely as a possible spur to discussion that I venture to give my own opinions in this very superficial review of work on the dynamics of homogeneous turbulence.

Speaking very broadly, the basic dynamical problem has been in the doldrums, to some extent, during the last 10 years, and has been in need of good ideas and new lines of approach. Formal mathematical investigations have produced remarkably little of value; successful theoretical work has more often taken the form of simple deductions from an assumed plausible physical model of a limited aspect of the flow. Work on the dynamics of the energy-containing eddies has more-or-less run its course, for the reasons I have mentioned. A number of general procedures for calculation of various dynamical aspects of homogeneous turbulence have been devised, but none of them impresses me as being likely either to advance our understanding of turbulence or to achieve results on which we can place reliance. The universal similarity theory of the small-scale components of the motion stands out in this rather grey picture as a valuable contribution, of which an increasing number of applications is being made, especially in problems involving convection and diffusion of scalar or vector properties of the fluid. There is a need for two kinds of further work in connection with this theory; first, for some mathematical proofs of some of the notions of the theory, such as the increasing degree of disorder accompanying transfer of energy to larger wave-numbers, and second, for measurement of the mean values of third and fourth powers of velocity derivatives at very high Reynolds number. The latter measurements might also throw light on the obscure and interesting question of the way in which the energy of the small-scale components is distributed over the fluid.

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RELATIONSHIPS AMONG SOME DEDUCTIVE THEORIES OF TURBULENCE

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SOMMAIRE

Une brève discussion qualitative est donnée de quelque parenté entre plusieurs méthodes déductives de la théorie de la turbulence isotrope, qui ont progressé dans la dernière décade. Ainsi de la formulation de l'équation fonctionelle de HOPF, développement des moments des vitesses en puissances d'un nombre de Reynolds caractéristique, l'approximation de quasi normalite de PROUDMAN et REID et TATSUMI, et les développements des perturbations modifiées de KRAICHNAN et WYLD.

SUMMARY

A brief and qualitative discussion is given of some relationships among several deductive approachs to isotropic turbulence theory which have been advanced in the past decade. Included are HOPF's functional equation formulation, expansion of velocity moments in powers of a characteristic REYNOLDS number, the suasi-normality approximation of PROUDMAN and REID and TATSUMI, and the modified perturbation expansions of KRAICHNAN and WYLD.

In the past decade there have been a number of attempts to derive some of the properties of isotropic turbulence by deductive mathematical procedures, with the Navier-Stokes equations of motion as the starting point. The present talk is intended to bring out some similarities and differences among several of these attempts. We shall include Hopf's functional-equation formulation [6, 7], expansion of velocity-moments in powers of a characteristic Reynolds number [4], the quasi-normality approximation as used by PROUDMAN and REID [14] and TATSUMI [15, 16], and the modified perturbation expansions of KRAICHNAN [8-10] and WYLD [17]. The treatment will be brief and qualitative. A more extended presentation of some of the material to be covered has been given elsewhere [11].

For simplicity, let us consider the following idealized problem. A statistically homogeneous and isotropic ensemble of incompressible flows is established with a velocity distribution which is multi-variate normal at the initial instant. The initial distribution thus is completely specified by the initial velocity covariance tensor [1]. Let the initial state be characterized by the Reynolds number

$$R_0 = \frac{v_0 l_0}{\nu}$$

where v_0 is the r.m.s. velocity in any direction, l_0 is a suitably defined macroscale (length-scale of energy-containing eddies), and ν is the kinematic viscosity. It is desired to find the velocity moments at later times as the flows evolve according to the Navier-Stokes equation. In particular, it is desired to follow the evolution of the velocity covariance tensor and the associated wavenumber-spectrum of kinetic energy.

This problem leads to coupled equations of motion for the velocity moments of all orders [1]. The coupling is due to the nonlinearity of the Navier-Stokes equation. The various attempts at deductive theories of isotropic-turbulence dynamics have been concerned largely with finding mathematically plausible and well-defined approximations which reduce the infinite set of coupled moment equations to a closed, finite set that yields physically acceptable solutions. The successes which have been achieved are very modest. Severe theoretical difficulties are posed by the combination of strong nonlinearity and strong dissipation which characterizes the problem for $R_0 \gg 1$. The strong nonlinearity makes perturbation expansions of usual type inappropriate, and the strong dissipation precludes use of the techniques of equilibrium statistical mechanics.

Hopf [6, 7] has reformulated a particular infinite subset of the complete set of coupled moment equations as a single functional equation for a characteristic functional of the velocity distribution. The moments involved are those which have simultaneous time-arguments. The reformulation does not in itself constitute progress toward a solution of the equations. However, it suggests several successive-approximation schemes, in particular the expansions in powers of R_0 and in cumulants which we shall now discuss. It should be noted that Hopf's formulation may be generalized so as to include the moments with non-simultaneous time-arguments.

A straightforward method of generating approximate solutions for the velocity moments is to expand them in powers of R_0, after transforming to appropriate dimensionless variables [11]. The expansion for the velocity covariance tensor is found to contain all even powers of R_0. The coefficient of R_0^{2n} is a linear functional of $(2n + 2)$-th order moments of the zeroth-order, viscous decay field. The latter field is that obtained by retaining only the linear terms in the Navier-Stokes equation; it is a linear functional of the initial velocity field and thus is normally distributed. It follows that the coefficients in the expansion of the exact velocity covariance tensor can all be expressed in terms of the covariance tensor of the viscous-decay field by use of the rules for evaluating higher moments of a normal distribution [1].

The difficulty with the power-series expansion is that it may be expected to converge well, if it converges at all, only for $R_0 < 1$, whereas the case of greatest interest is $R_0 \gg 1$. Convergence has not been demonstrated even for $R_0 \ll 1$, but it is plausible that the expansion is at least asymptotic in this range.

The expansion of the velocity covariance in powers of R_0 corresponds to an expansion of Hopf's characteristic functional in a functional power series [7]. Truncation of the R_0-expansion after some finite power corresponds to truncation of the functional series. This fact suggests the nature of the errors which can arise from using truncations of the R_0-expansion as approximations for the velocity moments : The truncated characteristic functional corresponds, in general, to a probability distribution which assigns negative probabilities to certain flows. A consequence is that the approximations can lead to negative kinetic-energy wavenumber-spectra $E(k)$ for some values of k. It is not difficult to verify that this actually occurs at high R_0. The negative values of $E(k)$ typically arise in the spectrum region which has the strongest initial excitation (initial energy-containing range). Thus the power-series approximations lead to qualitatively unacceptable consequences. This difficulty does not arise, however, if R_0 is very small, or if the time of evolution is very small.

A second approximation scheme suggested by Hopf's functional equation formulation consists of successive truncations of an expansion of the velocity moments into cumulants. This expansion corresponds to expansion of the logarithm of Hopf's characteristic functional in a functional power-series. The truncations yield closed sets of dynamical equations which involve only moments below a finite order. The first nontrivial approximation is obtained by discarding the fourth-order cumulants. This yields the quasi-normality approximation proposed by Millionstchikov [12] and exploited by Proudman and Reid [14], Tatsumi [15, 16], and others.

There is a mathematical relation between the approximations to the velocity covariance (or spectrum function) obtained by truncating the R_0-expansion and those obtained by truncating the cumulant-expansions. Suppose that all cumulants above order $n+2$ are discarded and the resulting dynamical equations are solved to yield the velocity covariance. The resulting approximate expression may be expanded in powers of R_0, and this series may be compared with the full, formally exact power-series expansion of the covariance. One finds that the two expansions agree precisely up to and including order R_0^{2n}. In each higher (even) order, the cumulant-discard result contains a non-vanishing contribution, but this contribution consists of only some of the terms from the exact expansion. The successive cumulant-discard approximations may be regarded mathematically as successively more comprehensive partial summations of all orders of the formally exact power-series expansion for the covariance tensor.

As the preceding paragraph suggests, for very small R_0 the successive cumulant-discard approximations are indistinguishable from the successive power-series truncations. This is also true for any R_0, provided the time of evolution is short enough. The fact that each cumulant-discard approximation, even the lowest, contains all powers of R_0 raises the hope that these approximations may remain valid for large R_0 and times which are not short. Unfortunately, this does not appear to be the case. The domains of validity of the cumulant-discard approximations and the simpler power-series approximations appear to be essentially the same, and outside the domain of validity both sets of approximations give rise to the physically unacceptable phenomenon of negative values for $E(k)$. Negative values of $E(k)$ arise in the cumulant-discard approximations because, as in the previous case, the truncated characteristic functional implies negative probabilities.

The serious negative probability troubles which arise in the cumulant-discard approximations have recently been demonstrated by several authors. Betchov [2] has found that the vorticity equation obtained by Proudman and Reid [14] is inconsistent with a positive-definite probability distribution. Ogura [13] has found by direct numerical integration that the spectrum equations of Proudman and Reid [14] and Tatsu-

MI [15] yield negative values of E (k). As in the case of the power-series approximations, the negative values of E (k) occur in the spectrum region which constitutes the initial energy-containing range, and they are comparable in magnitude with the initial spectrum level in this range. FRANCIS and O'BRIEN [5] have found that analogous results occur when the quasi-normality approximation is applied to the convection of a scalar field by isotropic turbulence. The unphysical results associated with negative probabilities persist in the higher cumulant-discard approximations [11].

An approximation scheme based upon a sequence of modified perturbation expansions has been proposed by the present author [8-10]. The approximations are dynamical in nature and are not made directly upon the probability distribution. The scheme involves the velocity moments for non-simultaneous as well as simultaneous time-arguments. In addition, it involves, in essential fashion, the functions which give the mean response to infinitesimal external perturbations of the amplitudes of the various spatial Fourier modes. Thus, the scheme cannot be represented within the present framework of HOPF's formulation.

In the lowest or direct-interaction approximation, the triple correlation of the amplitudes of each triad of spatial Fourier modes is replaced by the contribution to this correlation which is induced by the direct dynamical interaction of the three modes, acting against the relaxation effects produced both by viscosity and by the coupling of each of the three modes to all the rest of the Fourier modes. The result is a pair of closed equations which determine the velocity covariance tensor and the response functions of the Fourier modes.

This approximation differs from the first nontrivial truncation of the R_0-expansion for the triple correlation in that the latter approximation corresponds to retaining only the relaxation effect of viscosity. Crudely speaking, we may say that the additional effects retained in the direct-interaction approximation represent the action of " eddy-viscosity " upon the individual Fourier amplitudes. In the energy-containing and inertial ranges at high R_0, the direct effects of ordinary viscosity are negligible compared to those of eddy-viscosity. The two approximations differ profoundly in these ranges, except at very short evolution times, when they are indistinguishable.

If the direct-interaction approximation for the velocity covariance tensor, or spectrum function, is expanded in powers of R_0, we find that, like the quasi-normality approximation, the result agrees precisely with the exact expansion up to and including the terms in R_0^2. All higher (even) powers of R_0 are present also, and the coefficients of the higher powers again correspond to certain terms from the coefficients in the exact expansion. Like the quasi-normality approximation, the direct-interaction approximation may be considered an infinite partial summation of the exact expansion. However, the two approximations represent very different partial summations. The higher powers of R_0 in the direct-interaction result represent the eddy-viscosity effects mentioned in the preceding paragraph. These effects are not included in the results of Proudman and Reid and Tatsumi, as may be seen by examining the energy-transfer expression obtained by these authors [8]. For $R_0 \ll 1$, or for sufficiently short evolution times at higher R_0, the eddy-viscosity effects are negligible, and in this case the quasi-normality approximation and the direct-interaction approximation are both indistinguishable from the first power-series truncation. For high R_0, and times which are not short, the quasi-normality and direct-interaction approximations lead to very different results.

The direct-interaction approximation can be interpreted as an exact description of the behavior of a dynamical model [9, 10]. A consequence is that negative values of E (k), which characterize the power-series truncations and cumulant-discard approximations, cannot occur. It has not been proved that this holds also for the higher approximations in the present scheme. However, the second approximation has been investigated in part, and the results indicate that negative probability troubles in fact do not arise [10]. The second and higher approximations are defined by more elaborate dynamical approximations. They correspond to successively more comprehensive partial summations of contributions from all orders of the exact expansion in powers of R_0.

Wyld [17] has recently presented a formulation of the isotropic turbulence problem which follows closely certain techniques that have been used to investigate the renormalization problem of quantum electrodynamics. Like the schemes already discussed, the resulting formalism leads to infinite partial summations of expansions of quantities in powers of R_0. Wyld's equations involve the velocity covariance with non-simultaneous arguments and the average response functions which occur in the direct-interaction approximation. In addition, Wyld employs certain functions suggested by the vertex operators of quantum electrodynamics. These functions correspond to certain higher response functions that arise in the higher approximations cited in the preceding paragraph.

Wyld is able to represent in his formalism both the approximation of Chandrasekhar [3], which is based on an extension of the quasi-normality approximation to moments with non-simultaneous arguments, and the direct-interaction approximation. However, when so transcribed, neither approximation seems to have as clear a physical motivation as it did before. The original statistical or dynamical approximation is replaced by an abstract recipe for collecting infinite sub-sets of perturbation-theory terms.

The hope offered by Wyld's formulation is that it may suggest useful higher approximations which differ from those of the previous schemes. In this connection, however, the lack of a clear physical interpretation of Wyld's groupings of terms from the power-series expansions raises serious apprehension. We have seen that two of the approximation schemes examined in this talk lead to negative probabilities and negative values of E (k). This shows that the turbulence problem is unpleasantly sensitive to precisely what approximation is made. It is substantially more sensitive than quantum electrodynamics, which is essentially a weak-coupling problem. There seems to be little indication in Wyld's formulation as to whether the higher approximations to which it may lead will be free of negative spectrum troubles or not. Meanwhile, it is disconcerting that Chandrasekhar's approximation, which gives serious unphysical consequences [8], fits naturally into Wyld's scheme.

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APPENDIX

Note : This Appendix represents a consolidation and elaboration of remarks made by the author in the course of informal discussions during the Session.

Dr. PROUDMAN has raised the question of how similar are the quasi-normality and direct-interaction approximations. I should like here to extend the remarks made on this subject in my talk.

The differences in the assumptions underlying the two approximations have been mentioned in the text. The quasi-normality approximation involves neglect of fourth-order cumulants. The direct-interaction is a dynamical approximation and says nothing directly about what cumulants survive. Suppose that the Navier-Stokes equation is used to eliminate third-order moments in the equation of motion for the simultaneous-argument velocity covariance and replace them by expressions involving fourth-order moments. The latter may be written as the sum of a normal part and a part consisting of cumulants. In the quasi-normality approximation, as used by PROUDMAN and REID [14] and TATSUMI [15], the normal part is retained and the cumulants are put equal to zero. The direct-interaction approximation corresponds to retaining the normal part and giving the cumulants *non-zero values* determined by the dynamical approximation for the eliminated third-order moments. This approximation does not correspond to putting cumulants of any order equal to zero in the equations of motion for the velocity moments.

It should also be noted that the equations of the direct-interaction approximation involve, in essential fashion, the average response functions of the Fourier modes. These quantities do not enter into the quasi-normality equations.

So far as I have been able to discover, the extent of the relation between the quasi-normality and direct-interaction approximations in the problem at hand is that they both agree, through terms in R_0^2, with the formally exact expansion of the velocity covariance tensor in powers of R_0. This means that for very small times or small R_0 they give indistinguishable results. In this case, however, neither approximation offers any advantage over the simpler approximation of truncating the expansion of the covariance tensor after the terms in R_0^2; all three approximations give results which differ inappreciably.

For large R_0 and times which are not very short, the case in which the quasi-normality and direct-interaction approximations are interesting, they differ profoundly. The quasi-normality approximation, like simple truncation of the R_0-expansion, leads to large negative values of $E(k)$. The direct-interaction approximation cannot give negative $E(k)$. Ogura's results [13] show that the negative values first occur in the spectrum region which comprises the energy-containing range of the initial velocity distribution. The consequences of the quasi-normality approximation therefore appear to be qualitatively inadmissible on physical grounds in the very spectrum region where originally it was hoped the approximation might be most appropriate.

The fact that quasi-normality leads to negative spectra and the direct-interaction approximation does not is associated with a fundamental and easily described difference in the energy-transfer functions given by the two approximations. As discussed in [8], the direct-interaction energy-transfer function acts in the direction of producing equipartition among the various wave-number modes. If at any time the excitation in a given mode is zero, the approximation yields a positive energy-flow into that mode by nonlinear transfer from the other modes. This situation is physically plausible and resembles many other cases in statistical physics.

A very different situation obtains in the quasi-normality approximation. The difference is particularly clear in the spectrum regions where the direct effects of viscosity are negligible. There we find that the quasi-normality approximation does not yield a transfer of energy which tends in general to produce equipartition. Rather than the rate of energy-transfer itself, *the time-derivative of the rate of energy-transfer vanishes in equipartition.* Thus the energy transfer function has a long memory : If it is negative in a strongly excited spectrum region (as in the energy-containing range in the initial period of decay), it remains negative in that region *after* the time the modes therein have fallen *below* equipartition with the initially more weakly excited neighboring modes. Consequently, the energy-spectrum exhibits an unphysical overshoot behavior, and it is this which leads to the negative spectrum regions which Ogura has demonstrated.

If one assumes that a steady-state of a certain sort has been achieved, one can extract from the quasi-normality approximation an " inertial-range " law which goes either as k^{-1} or as k^{-2}, depending on just what assumptions are made [8, 16]. Ogura's calculations suggest that these results are illusory and that, instead of reaching a quasi-equilibrium, the spectrum will evolve from prescribed initial values to a pathological form which does not bear even a qualitative resemblance to physically occurring spectra. Recent work by the present author (involving nonlinear systems with finite numbers of degrees of freedom) suggests that the quasi-normality approximation at high R_0 may actually lead to *infinite* negative values for $E(k)$ within finite times after the initial instant. If so, this raises the possibility that the vorticity singularity at finite time, which PROUDMANN and REID [14] obtained for infinite R_0, may be associated with such behavior.

In contrast, the direct-interaction approximation leads to a non-pathological inertial range in which energy-transfer proceeds by a dynamically stable local cascade process [8]. The asymptotic inertial range law has the form $k^{-3/2}$, which differs only slightly form the generally accepted Kolmogorov $k^{-5/3}$ law. The difference in power law actually represents an important departure in dynamics from that called for by the Kolmogorov thoory. Moreover, it can lead to large quantitative errors in the absolute spectrum level in the inertial range at sufficiently high R_0. Nevertheless, the direct-interaction approximation appears to represent the only deductive theory published which really gives an inertial range at all. The self-consistency of the approximation permits the hope that it represents the first in a convergent sequence of successive approximations. Work on the second approximation (mentioned in the text) is at an early stage, but results to date suggest that it will reduce by an order-of-magnitude the quantitative error of the direct-interaction approximation in the inertial range. Numerical results which have already been obtained in the simpler problem of turbulent dispersion give additional support to this indication. Both the direct-interaction approximation and the second approximation have been applied to the determination of the probability distribution for the displacement of a fluid particle convected by isotropic turbulence, with neglect of molecular diffusion. In the second approximation, the errors are typically reduced by a factor of over ten [11].

In the energy-containing range, it is possible that the direct-interaction approximation itself may give acceptable quantitative approximations to the spectrum. Application of the approximation to nonlinear systems with finite numbers of degrees of freedom has given some support to this hope. It is possible that the approximation may be quantitatively adequate for investigating shear-turbulence and thermal-turbulence problems where the inertial range either is non-existent or plays a minor role in the dynamics.

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ON KRAICHNAN'S THEORY OF TURBULENCE

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SOMMAIRE

Cette communication est une brève revue des fondements cinématiques et dynamiques de la théorie de KRAICHNAN pour la turbulence homogène. Dans la partie cinématique, l'hypothèse de KRAICHNAN de « faible dépendance » est reprise en termes de représentation de la turbulence par l'intégral de FOURIER, et la voie par laquelle cette « hypothèse » est une conséquence de l'homogénéité est indiquée. Dans la partie dynamique, la connexion générale entre l'hypothèse de KRAICHNAN « d'interaction directe » et les théories dans lesquelles on suppose nuls les cumulants du quatrième ordre, est examinée. Il est suggéré que la théorie de KRAICHNAN a les mêmes défauts très généraux que les théories aux cumulants du quatrième ordre nuls, et qu'il est peu vraisemblable qu'elle conduise à des résultats utiles — spécialement pour les plus petits tourbillons, pour lesquels elle conviendrait théoriquement.

SUMMARY

This paper is a brief review of the kinematical and dynamical foundations of KRAICHNAN's theory of homogeneous turbulence. In the kinematical part, KRAICHNAN's « weak dependence hypothesis » is recast in terms of the FOURIER integral representation of turbulence, and the way in which this « hypothesis » is a consequence of homogeneity is indicated. In the dynamical part, the general connection between KRAICHNAN's « direct interaction hypothesis » and the zero-fourth-cumulant theories is examined. It is suggested that KRAICHNAN's theory has the same very general faults as the zero-fourth-cumulant theories, and that it is not likely to lead to useful results — especially for the smaller eddies to which it is conceptually most suited.

KRAICHNAN's analysis of homogeneous turbulence (see, for example, KRAICHNAN, 1959) is cast mainly in terms of the Fourier coefficients of the velocity field. The turbulent field is supposed to be spatially periodic with very large period L, and the velocity may then be expressed as a Fourier series

$$\mathbf{u}(\mathbf{x}, t) = \sum_{\mathbf{k}} \mathbf{A}(\mathbf{k}, t)\, e^{i\mathbf{k}.\mathbf{x}}, \tag{1}$$

where the summation is over all wave numbers permitted by the cyclic boundary conditions. In this talk, however, I shall use the Fourier integral representation (see

Batchelor, 1953) in which the large length necessarily introduced in the setting up of any Fourier representation of a stationary random function is used to define a box outside of which the velocity is taken to be zero. In this case the function corresponding to $\mathbf{A}(\mathbf{k}, t)$ in (1) is

$$\mathbf{B}(\mathbf{k}, t, \mathrm{L}) = \left(\frac{1}{2\pi}\right)^3 \int \mathbf{u}(\mathbf{x}, t, \mathrm{L})\, e^{-i\mathbf{k}.\mathbf{x}}\, d\mathbf{x}\,, \tag{2}$$

where the integration is over all space, and the integral of $\mathbf{B}$ over a finite region of $\mathbf{k}$-space is known to have a finite limit as $\mathrm{L} \to \infty$. Thus

$$\mathbf{dZ}(\mathbf{k}, t) = \lim_{\mathrm{L}\to\infty} \int_{\mathrm{V}(\mathbf{k})} \mathbf{B}(\mathbf{k}, t, \mathrm{L})\, d\mathbf{k}\,,$$

where $\mathrm{V}(\mathbf{k})$ is a volume element surrounding $\mathbf{k}$, and

$$\mathbf{dZ}(\mathbf{k}, t) = \frac{1}{(2\pi)^3} \int \mathbf{u}(\mathbf{x}, t)\, e^{-i\mathbf{k}.\mathbf{x}} \left(\frac{e^{-idk_1x_1} - 1}{-ix_1}\right) \left(\frac{e^{-idk_2x_2} - 1}{-ix_2}\right) \left(\frac{e^{-idk_3x_3} - 1}{-ix_3}\right) \tag{3}$$

when $\mathbf{k}$ and $\mathbf{k} + dk$ define opposite vertices of a small parallelepipedal element $\mathrm{V}(\mathbf{k})$.

The inverse relation, corresponding to (1), is then

$$\mathbf{u}(\mathrm{x}, t) = \int e^{i\mathbf{k}.\mathbf{x}}\, d\mathbf{Z}(\mathbf{k}, t). \tag{4}$$

While there are certain formal advantages in developing an analysis which is independant of the scale L, there is no essential physical distinction between (1) and (4), and I am not recasting Kraichnan's analysis in terms of an integral representation out of any feeling that the approach is superior. My hope is that the alternative presentation will itself be a contribution to a discussion of the theory.

One purpose for which the integral representation is especially useful is the estimation of the order of magnitude of the statistical moments of the distribution of $\mathbf{dZ}(k)$. It follows from (3) that any mean value of the form

$$\mathbf{dZ}_1\, \mathbf{dZ}_2\, \mathbf{dZ}_3 \ldots \tag{5}$$

is zero unless it is possible to choose wave numbers $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \ldots$, one from each of the volume elements in wave number space defining the $\mathbf{dZ}$'s, such that

$$\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \ldots = 0\,. \tag{6}$$

When this condition is satisfied, the magnitude of the mean value depends on the magnitudes of the wave-number volume elements. The simplest way to proceed, for order-of-magnitude purposes, is to suppose that all the volume elements are of the same small order V and that the geometry is not so special that the order of magnitude of the mean value is reduced from the maximum value for such elements. One then finds that

$$\overline{\mathbf{dZ}_1\, \mathbf{dZ}_2} = 0(\mathrm{V})\,, \tag{7}$$

$$\overline{\mathbf{dZ}_1\, \mathbf{dZ}_2\, \mathbf{dZ}_3} = 0(\mathrm{V}^2) \tag{8}$$

and that for higher moments it is the cumulants of the distribution that continue the series. Thus

$$\overline{\mathbf{dZ}_1\, \mathbf{dZ}_2\, \mathbf{dZ}_3\, \mathbf{dZ}_4} - \overline{\mathbf{dZ}_1\, \mathbf{dZ}_2} \cdot \overline{\mathbf{dZ}_3\, \mathbf{dZ}_4} - \ldots = 0\,(\mathrm{V}^3)\,, \tag{9}$$

$$\overline{\mathbf{dZ}_1\, \mathbf{dZ}_2\, \mathbf{dZ}_3\, \mathbf{dZ}_4\, \mathbf{dZ}_5} - \overline{\mathbf{dZ}_1\, \mathbf{dZ}_2} \cdot \overline{\mathbf{dZ}_3\, \mathbf{dZ}_4\, \mathbf{dZ}_5} - \ldots = 0\,(\mathrm{V}^4)\,, \tag{10}$$

etc.

The results are equivalent to KRAICHNAN'S *weak dependence hypothesis,* that the Fourier coefficients, when normalized with the root-mean-square magnitude $V^{1/2}$, have a distribution which trends, as $V \rightarrow 0$, to one representing statistical independence for distinct wave numbers (i. e. wawe numbers $\mathbf{k}_1$, $\mathbf{k}_2$ such that $\mathbf{k}_1 \neq \pm \mathbf{k}_2$). I am not convinced that such a normalization is of much significance for the odd-order moments, but there are in any case significant relations here. In (9), for instance, if the four wave numbers are such that

$$\mathbf{k}_1 + \mathbf{k}_2 = 0 \quad \text{and} \quad \mathbf{k}_3 + \mathbf{k}_4 = 0\,,$$

then

$$\overline{\mathbf{dZ}_1\,\mathbf{dZ}_2\,\mathbf{dZ}_3\,\mathbf{dZ}_4} \rightarrow \overline{\mathbf{dZ}_1\,\mathbf{dZ}_2} \cdot \overline{\mathbf{dZ}_3\,\mathbf{dZ}_4} \tag{11}$$

as $V \rightarrow 0$. Results of a statistically independent type, like (11), are implicit in any Fourier analysis of homogeneous turbulence. But it is curious, as KRAICHNAN pointed out in his earlier papers, that such results have not been emphasized more strongly in the literature. Probably the reason is that, without further ideas, these results are completely unusable in the dynaminal theory (as, of course, one would expect since they are merely consequences of homogeneity).

The formal way in which these results enter the dynamics is as follows. The dynamical equation for a Fourier coefficient is

$$\frac{\partial}{\partial t_1}\,\mathbf{dZ}_1 = i\int_{\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}_1}\left(\frac{\mathbf{k}_1\,\mathbf{k}_1\cdot\mathbf{dZ}_2}{\mathbf{k}_1^2} - \mathbf{dZ}_2\right)\mathbf{k}_1\cdot\mathbf{dZ}_3 - \nu\mathbf{k}_1^2\,\mathbf{dZ}_1$$

$$= \mathbf{P}_1\int_{\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}_1}\mathbf{dZ}_2\,\mathbf{dZ}_3 - \nu\mathbf{k}_1^2\,\mathbf{dZ}_1 \tag{12}$$

where P_1 is a third-order tensor depending only on $\mathbf{k}_1$. A typical dynamical equation for a mean value is therefore

$$\frac{\partial}{\partial t_1}\overline{\mathbf{dZ}_1\,\mathbf{dZ}_4\,\mathbf{dZ}_5} = \mathbf{P}_1\int_{\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}_1}\overline{\mathbf{dZ}_2\,\mathbf{dZ}_3\,\mathbf{dZ}_4\,\mathbf{dZ}_5} - \nu\mathbf{k}_1^2\,\overline{\mathbf{dZ}_1\,\mathbf{dZ}_4\,\mathbf{dZ}_5} \tag{13}$$

where the time differentiation is partially with respect to the time t_1 at which the Fourier coefficient $\mathbf{dZ}_1$ is taken. The integrand of the inertial interaction integral is then of different order of magnitude according as the wave numbers $\mathbf{k}_2$, $\mathbf{k}_3$, $\mathbf{k}_4$, $\mathbf{k}_5$ are or are not equal and opposite in pairs. Thus

$$\int\overline{\mathbf{dZ}_2\,\mathbf{dZ}_3\,\mathbf{dZ}_4\,\mathbf{dZ}_5} = 2\,\overline{\mathbf{dZ}\,(-\mathbf{k}_4)\,\mathbf{dZ}\,(\mathbf{k}_4)}\cdot\overline{\mathbf{dZ}\,(-\mathbf{k}_5)\,\mathbf{dZ}\,(\mathbf{k}_5)}$$

$$+\int(4^{\text{ht}}\text{ cumulant of }\mathbf{dZ}_2, \mathbf{dZ}_3, \mathbf{dZ}_4, \mathbf{dZ}_5), \tag{14}$$

with $\mathbf{k}_1 + \mathbf{k}_4 + \mathbf{k}_5 = 0$. The first term on the right is $0(V^2)$; the second term, being an integral over an integrand of $0\,(V^3)$, is $0\,(V^2)$; and, of course,

$$\overline{\mathbf{dZ}_1\,\mathbf{dZ}_4\,\mathbf{dZ}_5} = 0\,(V^2)\,.$$

So there are comparable contributions to the rate of change of a mean value from the discrete wave numbers where the statistical connection is strong and the totality of remaining wave-numbers where the statistical connection is weak. This is the sense in which the weak dependence 'hypothesis' is, alone, useless.

Nevertheless, most analytical attempts to discuss turbulence dynamics have made much of the distinction between the two types of term in (14). The zero-4th-cumulant-theories, for instance, neglect altogether the integral in (14); the corresponding form of (13), together with the dynamical relation between 2nd and 3rd order moments, forms a closed system of equations for the problem. Now the non-integral terms in (14) arise from particular terms in the original equation of motion for $\mathbf{dZ}_1$: out of the integration over all wave numbers in the inertia integral one retains only those wave numbers which appear in the other Fourier coefficients of the triple moment,

$$\frac{\partial}{\partial t_1}\mathbf{dZ}_1 = \mathbf{P}_1\,[\mathbf{dZ}\,(-\mathbf{k}_4)\,\mathbf{dZ}\,(-\mathbf{k}_5) + \mathbf{dZ}\,(-\mathbf{k}_5)\,\mathbf{dZ}\,(-\mathbf{k}_4)] - \nu\mathbf{k}_1^2\,\mathbf{dZ}_1 \qquad (15)$$

(with $\mathbf{k}_1 + \mathbf{k}_4 + \mathbf{k}_5 = 0$).

In other words, leaving aside any statistical dependence that may be induced by initial conditions or driving forces, the zero-fourth-cumulant theory implies that the triple moment is non-zero only on account of interaction between its own three wave-numbers. Such a theory, which in some way singles out the interaction between the particular wave numbers of a mean value as being more important than other interactions, may be termed a direct interaction theory *. KRAICHNAN's theory is of this kind, and down at this conceptual level, therefore, it is closely related to zero-fourth-cumulant theories. Indeed both theories tend to have the same very general properties and to stand or fall by similar criteria. Neither theory seems to depend on the Reynolds number (in any obvious way) for its success or otherwise. Both theories involve this dynamically very artificial distinction between direct and indirect interaction; the only real distinction between such interactions is the order-of-magnitude one arising from the kinematics, and this has already been seen to be without significance as for as the intrinsic mechanics of turbulence is concerned.

However, if one does accept the idea that there is something dynamically worthwile in the distinction between direct and indirect interaction, then there is more than one way of proceeding. The simplest representation of the idea is the zero-fourth-cumulant one represented by (15), and this is the equation that HEISENBERG took to solve for $\mathbf{dZ}_1$ for the purpose of substituting in $\overline{\mathbf{dZ}_1\,\mathbf{dZ}_4\,\mathbf{dZ}_5}$. The fact that the form of the equation varies according to the purpose for which $\mathbf{dZ}_1$ is needed (i. e., varying $\mathbf{dZ}_4$, $\mathbf{dZ}_5$) is merely a reflection of the fact that we are not dealing with exact properties of homogeneous turbulence. The point is not as serious as it may seem at first sight because the part that varies according to purpose is small compared with the other terms in the equation. Thus

$$\frac{\partial}{\partial t_1}\mathbf{dZ}_1 \quad \text{and} \quad \nu\mathbf{k}_1^2\,\mathbf{dZ}_1$$

are both $0(\mathrm{V}^{1/2})$, whereas $\mathbf{dZ}_4\,\mathbf{dZ}_5$ is $0(\mathrm{V})$. What *is* serious is that the dominant terms in the equation represent pure viscous decay, so that if the Reynolds number is not small the resulting computation of the triple moment must be hopelessly in error.

The correction of this particular defect is the most interesting part of KRAICHNAN's theory and what makes it a conceptually superior exploitation of the motion of dominant direct interaction. The idea is not to describe the process that leads to the establishment of triple moments but to describe the process that does not lead to triple moments. Thus,

* I am here using the term « direct interaction » in a more general sense than KRAICHNAN does.

in his direct interaction hypothesis, KRAICHNAN asserts that a hypothetical process (distinguished by primed Fourier coefficients) which satisfies the equation

$$\frac{\partial}{\partial t_1} \mathbf{dZ}_1' = \mathbf{P}_1 \int_{\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}_1} \mathbf{dZ}_2' \, \mathbf{dZ}_3' - \nu k_1^2 \, \mathbf{dZ}_1'$$
$$- \mathbf{P}_1 \left[\mathbf{dZ}'(-\mathbf{k}_4) \, \mathbf{dZ}'(-\mathbf{k}_5) + \mathbf{dZ}'(-\mathbf{k}_5) \, \mathbf{dZ}'(-\mathbf{k}_4) \right], \tag{16}$$

i. e., the exact equation less the direct interaction between $\mathbf{k}_1$, $\mathbf{k}_4$, $\mathbf{k}_5$, does not lead to the establishment of the triple moment $\overline{\mathbf{dZ}_1' \, \mathbf{dZ}_4' \, \mathbf{dZ}_5'}$. Thus the triple moment in the actual process envisaged in the KRAICHNAN theory arises only through replacement of the direct interaction terms in (16). In view of the non-linearity of the governing equation this inverse statement of the direct interaction concept is not equivalent to that used in the zero-fourth-cumulant theories. The dominant terms in (16) are those in the exact NAVIER-STOKES equation, and the response of the system to the small direct interaction terms is modulated by every FOURIER coefficient in the representation — not just the linear response described by (15).

In the mean-value equations (13) and (14) this modulation of the direct interaction is included in the fourth-cumulant integral, and KRAICHNAN's theory therefore partially takes into account this term. The fourth cumulant that *is* assumed to be zero in the theory is the one referring to the hypothetical process described by (16) — as may be seen by multiplying that equation by $\mathbf{dZ}_4' \, \mathbf{dZ}_5'$ and averaging. Thus the infinitesimal difference between the hypothetical and actual processes.

$$\Delta \mathbf{Z}_1 = \mathbf{dZ}_1' - \mathbf{dZ}_1 = 0(\mathrm{V})$$

gives rise not only to a triple moment but also to a fourth cumulant. At first sight one might think that this represents some fundamental improvement over the approximation of the zero-fourth-cumulant theories. But I think that this cannot really be so. The situation is symbolically represented by the quadratic expression

$$(A + B)^2 = A^2 + 2AB + B^2$$

in which A and B stand (loosely) for direct and indirect interaction, respectively. The zero-fourth-cumulant theories assert that we should retain only terms containing A only; KRAICHNAN's theory asserts that we should neglect only terms containing B only. If there *were* asymptotic conditions under which either theory were valid ($B \to 0$) then the other would be automatically valid. Thus the difference between the theories is essentially a measure of the error of them both.

Finally, it may be useful to add a word about the value of attempting expansions of which the theories so far discussed represent the leading term. Successive terms of these expansions may, for the present purpose, be regarded as taking into account successively more complicated indirect interactions.

For this, the significance of the leading term is all important. If there were some dynamical conditions for which the direct interaction hypothesis were asymptotically exact, then the expansion would be an asymptotic expansion about these conditions with an easily understandable significance. But I have been emphasising that no such conditions exist and that subsequent terms in the expansion are of the same order as the leading term. The practical utility of the expansion (or rather the first few terms of it — which would be all that would be reasonable to compute) then depends very much on the ratio of direct interactions to indirect interactions. If this is 0(1) it might be reasonable to expect useful accuracy from a few terms. But if this ratio is very small then the expansion is not of much interest.

Unfortunately, there seems to be some suggestion from the observational evidence that indirect interactions are actually dominant for those components of turbulence that are least correlated with the generating mechanism. If one takes the deviation of the flatness factor

$$\frac{\left(\frac{\partial^n u}{\partial x^n}\right)^4}{\left[\left(\frac{\partial^n u}{\partial x^n}\right)^2\right]^2}$$

from three as an order-of-magnitude measure of the importance of indirect interactions, then the fact that this increases with both n and Reynolds number does seem to suggest dominant indirect interactions. If this is right, the futility of attempting an expansion whose leading term represents direct interaction is obvious, and I do not hold out much hope for useful results from the theory.

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THE UNIVERSAL SMALL-SCALE SPECTRUM OF TURBULENCE AT HIGH REYNOLDS NUMBER

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SOMMAIRE

Des mesures récentes de GRANT, STEWART et MOLLIET apportent un soutien à l'idée d'une forme universelle du spectre de la turbulence pour la partie à petite échelle, aux grands nombres de REYNOLDS, mais les résultats des expériences du Jet Propulsion Laboratory sont différents, d'une manière significative.

Diverses théories de la fonction de transfert aux grands nombres d'ondes sont énumérées, et on trouve que leurs prévisions diffèrent peu les unes des autres. La comparaison avec les mesures de GRANT et coauteurs montre une bonne concordance.

SUMMARY

Recent measurements by GRANT, STEWART & MOLLIET give support for the idea of a universal form for the small scale end of the spectrum of turbulence at high REYNOLDS numbers, but the results of experiments at the Jet Propulsion Laboratory are significantly different.

Various theories for the transfer function at high wave numbers are listed, and it is found that their predictions differ little from each other. Comparison with the measurements by GRANT *et al.* shows fair agreement.

Introduction

When I was invited to present this paper, it was hoped that it would be possible to make a critical assessment of recent experiments which bear on the KOLMOGOROFF theory of the small scale end of the spectrum. Two principal series of measurements have been made, those of GRANT, STEWART and MOLLIET in a tidal current in the ocean and those at the Jet Propulsion Laboratory in grid turbulence at rather high Reynolds numbers. Unfortunately the results of the latter did not become available until the Colloquium was under way, when they were found to disagree rather strongly with those of GRANT *et al.* The nature of this disagreement will be described below, but it is too early to form any judgement on the accuracy of the experiments.

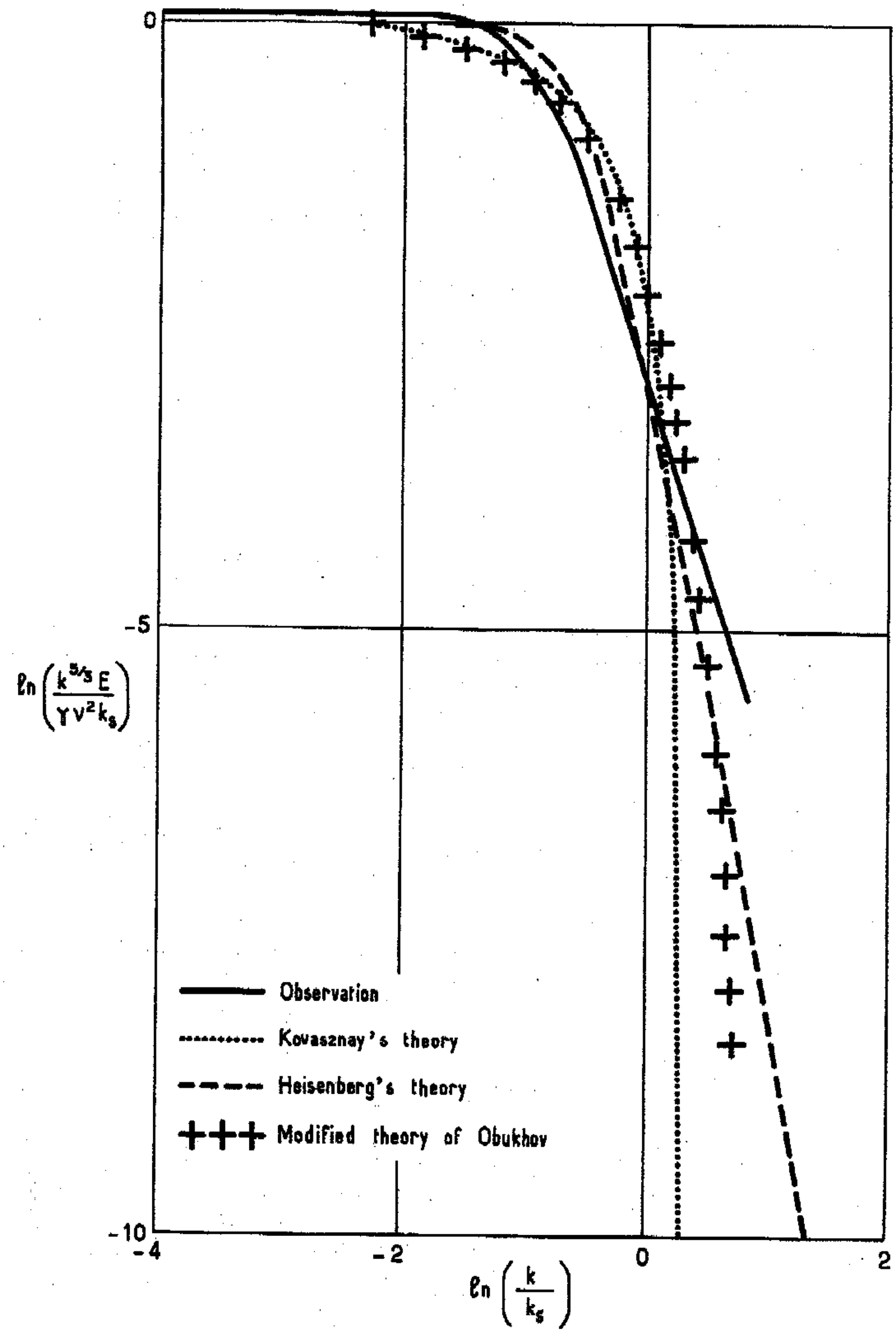

FIGURE 1
Comparison between theories and observed spectra transformed assuming isotropy.

Concept of Kolmogoroff's theory

The basic idea of the theory is that turbulent energy is generated in some certain range of length scale and then passed to smaller and smaller scales by a cascade process in which the steps are not too lage. Thus the smaller scales are only indirectly coupled to the larger ones; and, outside the range where creation is taking place, the energy transfer across any wave-number completely determines the statistical properties of the motion at smaller scales. Thus if we write $E(k)$ for the spectrum*, at high Reynolds numbers where the scales of creation and dissipation are well separated (and, if the turbulence is decaying $\frac{d\,ln\,E}{dt}$ is small compared with a charateristic frequency of the eddies on that scale), outside the range of scales where creation is important,

$$\begin{aligned} E &= E\,(k, \varepsilon, \nu) \\ &= \gamma \varepsilon^{2/3}\, k^{-5/3}\, F_1\,(k/k_s) \end{aligned} \tag{1}$$

where $k_s = \varepsilon^{1/4}\, \nu^{-3/4}$, on dimensional grounds.

The inertial sub-range

If one assumes further, as seems virtually certain, that viscosity does not enter into the transfer process at scales where the dissipation is negligible, then one would expect that in the range between the scales of creation and dissipition $F_1 = 1$, and so

$$E = \gamma \varepsilon^{2/3}\, k^{-5/3} \tag{2}$$

Stewart & Towsend (1951) have discussed the necessary conditions for the inertial range to be extensive in grid turbulence and find that a value of R_M of at least 10^6 is required. This is not easily attained in the laboratory (though measurements at the Jet Propulsion Laboratory which are not yet available come near it). For flows in channels we may assume that the criterion is $Lk_s \gg 10$. This implies $Lu'/\nu \gg 10^4$. This is more easily attained, but even so natural turbulence provides much the most readily available source.

Experiments of Grant, Stewart & Molliet (1962)

These observations, of which a full account is in course of publication **, provide what is probably by far the most convincing demonstration of the correctness of the Kolmogoroff theory that has yet been made. The measurements were made using hot film equipment in a tidal channel where the Reynolds number based on depth and mean velocity took values up to 3×10^8. They found that by plotting all their spectra in the correct dimensionless manner they could all be superposed to within the experimental accuracy; the power law in the inertial range was shown to lie within the range 1.66 to 1.75; and the value of γ was found to be 1.34 with a standard error of the mean

* A number of detailed definitions are given in the Appendix.

** A number of diagrams of the data were shown on slides at the Colloquium by Prof. R. W. Stewart.

of 0.05. The values of ε covered by the measurements range from 0.002 to 1. A comparison has been made between the measured spectra of $\left(\frac{du}{dx}\right)$ and those found by earlier workers in pipes and channels, and agreement is found to be quite as good as would be expected in view of the limited Reynolds number of the earlier experiments.

The practical importance of these observations is enormous since they enable the whole of the high frequency end of the spectrum in many natural situations to be determined from measurements at one frequency which are technically simple.

Experiments at Jet Propulsion Laboratory

These were made behind a grid in a pressurised wind-tunnel with values of R_M up to 2.4×10^6. It is not possible to make any assessment of them until a full account is published, but the results presented during the Colloquium * revealed several unexpected features :

(i) They indicate a value for γ of about 3.3.

(ii) Measurements were made of the cross component of the turbulence (which has not yet been done systematically by GRANT *et al.*) which show a marked lack of isotropy which depends very little on frequency. At the highest measured frequencies the spectrum of one cross component of velocity was a factor of about 1.7 lower than would be expected from the spectrum of u assuming isotropy.

If these conclusions are confirmed in the future, it is clear that the whole basis of the Kolmogoroff theory will be suspect and a radical revision of existing ideas in the mechanism of the transfer of turbulent energy through the spectrum will be called for.

Other estimates of γ

In Appendix various well known formulae of isotropic turbulence are listed. From these the relation of γ to the so-called structure functions and to the skewness of $\frac{du}{dx}$ may be seen. The present value of γ indicates a value of -0.34 ± 0.015 for the skewness of $\frac{du}{dx}$. This is in remarkably good agreement with the values measured by Twnsend and Stewart (BATCHELOR, 1953, p. 118) at much lower Reynolds numbers.

Atmospheric measurements

Spectra following the $-5/3$ power law have been found quite commonly in the atmosphere, but no reliable measurements of γ are available. It is however interesting to note that here, as in the ocean, the universal spectrum covers a vast range of scale. For example, ZBROZEK and RIDLAND (1960) give a spectrum measured at a height of 800 m with $U = 26\,\mathrm{ms}^{-1}$. If one uses the present value of γ, one finds $\varepsilon = 6.6$ c.g.s.

* Also on slides, by Dr. A. KISTLER.

and so $k_s = 6.7$ cm^{-1}. The measured 5/3 power law extends to scales of the order of 500 m and so the whole of the spectrum from this scale downwards could now be determined from measurements at one frequency.

Theories of the spectral form in the viscous sub-range

A brief summary of the better known theories which predict the form of the spectrum in the viscous range will now be given. None of these theories can be expected to apply when k is much greater than k_s and viscosity is dominant, but they are all reasonably successful near the end of the $-5/3$ law range and have no adjustable constants whatsoever.

Let us introduce the following dimensionless variables : $\kappa = \frac{k}{k_s}$, $\mathcal{E} = \gamma\kappa^{-2/3} F_1(\kappa)$, $\Sigma = \frac{S}{\varepsilon}$, where $S(k)$ is the rate at which energy is being transferred across wave-number k. Also, for convenience write $X = \kappa^{4/3}$, and $Y = \kappa^{2/3} \frac{\mathcal{E}}{\gamma}$.

The equation for the energy balance now takes the form

$$\kappa\mathcal{E} = \frac{1}{2} - \frac{d\Sigma}{d\kappa} \tag{3}$$

and one additional relation between the variables is needed to form a complete theory. The form of the relation depends on the physical content of the theory. We shall now briefly review the better know theories. First note the limiting conditions

(i) in inertial sub-range : $X = 0$; $Y = 1$; $\Sigma = 1$, and

(ii) at a large finite value of κ, or as κ tends to infinity, Σ and $\mathcal{E}$ must fall to zero together.

Kovasnay's theory

The rate of decay of turbulence is usually well represented by an expression of the form $\frac{v'^{3/2}}{L}$ where v' is a typical velocity and L a length scale. The idea behind Kovasnay's theory is that this formula applies to the transfer down the spectrum on any scale, so

$$S \propto k\,(kE)^{3/2}. \tag{4}$$

An equivalent derivation of this formula can be made from the assumption that S depends on the local values of E (in k-space) but not on the viscosity; so that the viscosity does not influence the transfer process directly at all. This may well be true near k_s, but does not seem very plausible at much higher wavenumbers .

The results of this and the other theories are summarized in Table 1. It will be seen that Kovasnay's theory leads to a sharp cut off in the spectrum at a finite value of κ.

Obukhov's theory

This suggests that the rate of transfer should be thought of as a Reynolds stress multiplied by a rate of strain due to the action of larger scales. This latter is taken to be equal to its r.m.s. value,

$$\left[2\int_0^k k'^2\,E\,(k')\,dk'\right]^{1/2}$$

and, in the original form of the theory, the Reynolds stress was taken as proportional to the intensity of all smaller eddies

$$\int_k^\infty E\,(k')\,dk.$$

Thus

$$S\,\alpha \int_k^\infty E\,dk' \left[2\int_0^k k'^2\,E\,dk'\right]^{1/2} \tag{5}$$

$$\Sigma = F\int_k^\infty k'^{-1}\,\mathcal{E}\,dk'\,(1-\Sigma)^{1/2} \tag{6}$$

This expression leads to a physically impossible form for the spectrum since Σ falls to zero at a value of κ where $\mathcal{E}$ is finite.

Modification of Obukhov's theory

The difficulty with this theory can be removed if one assumes that the Reynolds stress is produced only by eddies in the neighbourhood of k and not by all eddies equally, and so take it proportional to kE. This is plausible since the various scales of smaller eddies cannot be highly correlated one with another, but it inevitably brings the theory into close agreement with Kovasnay's since the integral for the rate of strain receives its greatest contribution from eddies in the neighbourhood of κ and the theory is almost a local one.

This time we obtain

$$\Sigma = F\,\mathcal{E}\,(1-\Sigma)^{1/2} \tag{7}$$

Heisenberg's theory

This starts from the observation that

$$\varepsilon = 2\,\nu\int_0^k k'^2\,E\,dk + S$$

so that if the transfer is thought to be due to the small eddies acting on the large ones as an eddy viscosity, one might expect S to be of the form

$$N\,(k)\int_0^k 2k'^2\,E\,dk'$$

Table I

Theory	Assumption	Non dimensional form	Resolt	Noteworthy features
Kovasnay	Local transfer independent of ν	$\Sigma = Y^{3/2}$	$Y = \left[1 - \frac{1}{2}\gamma X\right]^2$	Cutoff at $k_{lim} = \left(\frac{2}{\gamma}\right)^{3/4}$
Obukhov	$S \, \alpha \int_k^\infty E \, dk' \left[2 \int_0^k k'^2 E \, dk'\right]^{1/2}$	$\frac{\Sigma}{(1-\Sigma)^{1/2}} = F \int_k^\infty \kappa^{-1} \mathcal{E} \, d\kappa$	$\frac{F\kappa^{-2}}{2} = \frac{1 - \frac{1}{2}\Sigma}{(1-\Sigma)^{3/2}}$	E falles discontinuously to zero at $k_{lim} = \left(\frac{2}{F}\right)^{1/2}$ which is physically impossible.
Modified Obukhov .	$S \, \alpha \, Ek \left[2 \int_0^k k'^2 E \, dk'\right]^{1/2}$	$\frac{\Sigma}{(1-\Sigma)^{1/2}} = F \, \mathcal{E}$	$-\frac{\kappa^2}{F} = 2(1-\Sigma)^{1/2} + 2 \, \mathrm{In}\left(\frac{1-(1-\Sigma)^{/12}}{\Sigma^{1/2}}\right)$	$k \, E \, \alpha \, e^{k^2/F}$ as $k \to \infty$ $F = \left(\frac{2}{3\gamma^3}\right)^{1/2}$
Heisenberg	$S = H \int_k^\infty k'^{-3/2} E^{1/2} dk' \int_0^k 2 k'^2 E \, dk'$	$\frac{\Sigma}{1-\Sigma} = H \int_\kappa^\infty \kappa'^{-2} \mathcal{E}^{1/2} d\kappa'$	$Y = \left[1 + \frac{27}{8}\gamma^3 X^3\right]^{-4/3}$ $\Sigma = 1 - (1 - Y^{3/4})^{1/3}$	$E \, \alpha \, k^{-7}$ as $k \to \infty$ $H = \left(\frac{4}{3^{4/3}\gamma}\right)^{3/2}$

where N (k) is an eddy viscosity to be formed out of the properties of the small eddies. Assuming it to be an integral, dimensional considerations suggest

$$N(k) = H \int_k^\infty k'^{-3/2} E^{1/2} \, dk'$$

Thus we obtain

$$\frac{\Sigma}{1 - \Sigma} = H \int_\kappa^\infty \mathbf{k}'^{-2} \, \mathcal{E}^{1/2} \, d\kappa \tag{8}$$

In this theory as in the earlier ones the arbitrary constant can be expressed in terms of γ and when this is known from observations the theory becomes complete.

The predictions of the theories discussed above differ very little from each other in the vicinity of k_s, and are in moderate agreement with the observations. Heisenberg's theory in fact fits the observations best, but one may doubt whether this is significant. It is interesting to note that present value of γ leads to a value of H of 0.57 which may be compared with 0.45 ± 0.05 estimated by Proudman from consideration of the whole spectrum (Batchelor, 1953, p. 167).

Townsend's theory

At values of k much greater than k_s it is unlikely that any theories of the form just discussed will apply. It seems likely that a model on the lines of that of stretching vortex sheets put forward by Townsend is needed. Unfortunately, as pointed out by Batchelor in a preceding paper at this colloquium, this theory is not at present satisfactory since it leads to an infinite value for the mean-square vorticity.

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APPENDIX

DEFINITIONS AND USEFUL FORMULAE

Normalisation of spectra

$$\int_0^\infty \mathrm{E}(k)\, dk = \frac{1}{2}\,(u_1^2 + u_2^2 + u_3^2)$$

$$\int_0^\infty \varphi_{11}\, dk = u_1^2; \qquad \int_0^\infty \varphi_{22}\, dk = u_2^2$$

Isotropy relations

$$\mathrm{E} = \frac{1}{2} k^3 \frac{d}{dk}\left(k^{-1} \frac{d\,\varphi_{11}}{dk}\right)$$

$$\varphi_{11} = \int_{k_1}^\infty \left(1 - \frac{k_1^2}{k^2}\right) \mathrm{E}\, k^{-1}\, dk$$

$$\varphi_{22} = -\frac{1}{2} k^2 \frac{d}{dk}\,(k^{-1}\,\varphi_{11})$$

Dissipation

$$\varepsilon = 15\,\nu \int_0^\infty k^2\, \varphi\, dk$$

$$= 2\,\nu \int_0^\infty k^2\, \mathrm{E}\, dk$$

Characteristic quantities for small scales

$$k_s \equiv \varepsilon^{1/4}\, \nu^{-3/4}$$

$$v \equiv \varepsilon^{1/3}\, k_s{}^{-1/3}$$

$$= \varepsilon^{1/4}\, \nu^{1/4}$$

Dimensionless variables

$$\kappa \equiv \frac{k}{k_s}$$

$$\Sigma \equiv \frac{\mathrm{S}}{\varepsilon}$$

$$\mathcal{E} \equiv \frac{k\mathrm{E}}{v^2}$$

$$\mathrm{X} \equiv \kappa^{4/3}$$

$$\mathrm{Y} \equiv \kappa^{2/3}\, \gamma^{-1}\, \mathcal{E}$$

Energy balance

$$2\nu k^2\, \mathrm{E} = -\frac{dS}{dk}$$

or

$$\kappa \mathcal{E} = -\frac{1}{2} \frac{d\Sigma}{d\kappa}$$

$$\text{or}\qquad Y^{-1}\frac{d\Sigma}{dY}\frac{dV}{dX}=\frac{3}{2}-\gamma$$

Kolmogoroff relations

$$E=\gamma\varepsilon^{2/3}k^{-5/3}F_1\left(\frac{k}{k_s}\right)$$

$$\mathcal{E}=\gamma\kappa^{-2/3}F_1(\kappa)$$

In inertial subrange	*Exptl. values of coefficients*
$E=\gamma k^{-5/3}\varepsilon^{2/3}$	$\gamma=1.34$
$\varphi_{11}=\frac{18}{55}\gamma k^{-5/3}\varepsilon^{2/3}$	$0.3273\,\gamma=0.44$
$\Sigma=1$	
$\mathcal{E}=\gamma k^{-2/3}$	
$Y=1$	
$B_{dd}\equiv[u_1(x_1)-u_1(x_1+r_1)]^2=\frac{81}{55}\left(\frac{1}{3}\right)!\,\gamma r^{2/3}\varepsilon^{2/3}$	$1.315\,\gamma=1.76$
$B_{nn}\equiv[u_2(x_1)-u_2(x_1+r_1]^2=\frac{108}{55}\left(\frac{1}{3}\right)!\,\gamma r^{2/3}\varepsilon^{2/3}$	$1.754\,\gamma=2.35$
$\left(\frac{du_1}{dx_1}\right)^3\left[\left(\frac{du_1}{dx_1}\right)^2\right]^{-3/2}=-\frac{4}{5}\left[\frac{81}{55}\left(\frac{1}{3}\right)!\,\gamma\right]^{-3/2}$	0.34

DISCUSSION DE LA SECTION : TRANSFERT D'ÉNERGIE EN TURBULENCE HOMOGÈNE

G. K. BATCHELOR

Referring to BATCHELOR's suggestion about direct numerical integration of the (unaveraged) NAVIER-STOKES equation with respect to time, with a random initial distribution of velocity, CORRSIN said that, according to some estimates he had made (*American Scientist,* vol. 49, 1960, page 300), the memory required of a computing machine for such an integration is enormous and beyond the reach of existing machines. TAYLOR and LIEPMANN were of the opinion that some useful results might be computable for turbulence of moderate REYNOLDS number, although it was generally agreed that a calculation for high REYNOLDS number which revealed properties of the small-scale components would be more valuable.

At several points in the discussion, the need for more measurements of the kinetic energy spectrum and other quantities in turbulence at very high REYNOLDS number was emphasized. The available theories concerning the small-scale components of homogeneous turbulence are apparently applicable only at REYNOLDS numbers well above those at which most of the measurements have been made. KOVASZNAY suggested that the time has come for the investment of a large amount of money in some experimental facility which would make possible measurements of the small-scale components at really high REYNOLDS numbers.

With regard to the differences between the data on the spectrum of the u-component and that of the v-component obtained by KISTLER at J. P. L., LAUFER believed that they could not be explained by the fact that the velocity components u and v were measured with different instruments, viz. the single hot wire and the X-type hot wires. The people at J. P. L. had been concerned about this discrepancy, but could not detect any extraneous probe effects. In particular they had compared spectrum measurements with the single and X-wire probes in the plane-wave sound field of a turbulent boundary layer in which the u-fluctuations differ only by a constant from those of v. The normalized spectrum distributions obtained by the two probes were identical. LAUFER also mentioned that KLEBANOFF, working with a turbulent boundary layer, and he, working with the central region of turbulent pipe flow, had obtained similar forms of the v-spectra.

BETCHOV described some experimental techniques which may be useful in an investigation of the intermittency of high-order spatial derivatives of the velocity. From a given signal $f(t)$, one can determine both the power spectrum of f, to be denoted by $\varphi(\omega)$, and the power spectrum of f^2, to be denoted by $\Psi(\omega)$. It the FOURIER components of $f(t)$ have no phase relations, the ratio η defined as

$$\eta(\omega) = \frac{\Psi(\omega)}{\int_{-\infty}^{\infty} \varphi(k)\, \varphi(k+\omega) dk}$$

is equal to unity. BETCHOV had found that in grid turbulence, and with the acceleration $\frac{Du}{Dt}$ (where u = velocity) as the signal f, $\eta = 1$ for values of ω up to the KOLMOGOROFF scale and rises to 2 for large values of ω. It would be useful to repeat these measurements at larger REYNOLDS numbers.

This effect of phase relations is closely related to the existence of spikes in the distribution of $\frac{Du}{Dt}$. BETCHOV believed it would be interesting to study the size and shape of the region of space corresponding to a single spike. This could perhaps be done with several hot-wires, each signal being differentiated and passed through an electronic gate which delivers a signal of magnitude unity during a spike and zero otherwise. The gated signals could be applied to a coincidence counter, giving a *coincidence* function, instead of the usual correlation function.

KISTLER reported that some recent measurements at J. P. L. of the decay of grid turbulence seem to show a dependence of the decay law in the region near the grid ($\frac{x}{M}$ between 20 and 100) on the shape of the turbulence-producing elements. A grid was constructed of plastic spheres supported in a square array by a mesh of fine wires. Measurements of the decay of $\overline{u^2}$ or $\overline{v^2}$ showed that (1) the decay law for either $\overline{u^2}$ or $\overline{v^2}$ could be closely represented by $\overline{u^2} \propto \left(\frac{x}{M}\right)^{-4/3}$, and (2) the variation of the turbulent microscale as determined from the spectra was $\lambda^2 = 7.5\, \nu t$. This decay law is the same as would be expected in the wake of an isolated sphere after the wake attains some sort of an equilibrium structure. KISTLER pointed out that the usual initial period result, $\overline{u^2} \propto \left(\frac{x}{M}\right)^{-1}$, obtained behind essentially two-dimensional turbulence-producing elements, is the same as the law of development of the wake of an isolated rod.

UBEROI said that his measurements in the turbulence behind the usual biplane grids at $\frac{MU}{\nu} = 2.6 \times 10^4$ showed that $\overline{v^2} = \overline{w^2} = 0.7\, \overline{u^2}$ and that one-dimensional spectra of $\overline{u^2}$ and $\overline{v^2}$ do not satisfy the relation imposed by the requirement of isotropy. These unequal turbulent intensities imply that there is less vorticity along the mean flow than across it. Immediately behind the grid there is negligible vorticity along the mean flow and apparently the mixing further downstream is not strong enough to give complete isotropy. Following this idea, a grid of circular rods which make equal angles with all the axes was constructed. The turbulence produced by this grid was not noticeably more isotropic than that produced by usual grids. The grid rods were roughened to induce a turbulent boundary layer on them. This did not make the turbulence more nearly isotropic. The vorticity along the mean flow may also be increased by contracting the flow behind the grid. It was possible to make $\overline{u^2} = \overline{v^2} = \overline{w^2}$ but the spectra of $\overline{u^2}$ and $\overline{v^2}$ were not then related as in the isotropic case. Furthemore, in the uniform section behind the contraction $\overline{u^2}$ began to exceed $\overline{v^2}$.

Uberoi had found no finite range of wave-numbers over which the measured energy transfer is zero or small, and therefore no Kolmogoroff inertial subrange, in the spectrum of grid-generated turbulence at $\frac{UM}{\nu} = 2.6 \times 10^4$. Kistler confirmed (see the talk by Ellison) that anisotropy in grid-generated turbulence seems to persist at high wave-numbers at $\frac{UM}{\nu} = 2.4 \times 10^6$, and Uberoi concluded that an inertial subrange does not exist even at this high Reynolds number. In view of all this, he believed that the fact that the spectra of grid turbulence and shear-flow turbulence agree at high wave-numbers when plotted in terms of Kolmogoroff parameters should be interpreted with caution.

On the theoretical side, it was reported that O'Brien (*J. Fluid Mech.*, 12, 1962 — in the press) had used the zero fourth-order cumulant assumption as a means of calculating the development of the spectrum of a convected scalar quantity θ. The initial spectra of θ and of kinetic energy were so chosen as to have maxima near wave-number k_0. The fourth-order cumulants of the joint distributions of θ and the velocity were put equal to zero in the familiar way, and a forward integration of the equations with respect to time was carried out numerically. It was found that the spectrum of θ developed a local minimum, and became negative at a wave-number near k_0 after a time of order $\frac{k_0}{(\overline{u^2})^{1/2}}$.

It was also reported at the Colloquium that Golitsyn (*Prik. Mat. i Mech.*, vol. 24, 1960) had calculated the form of the energy spectrum at very large wave-numbers, on the assumption that the skewness factor of the difference between the velocities at two points is constant for all distances between the points small compared with the size of the energy-containing eddies; he found that after a certain time the spectral density becomes negative for some wave-numbers. It was remarked that this is effectively another hypothesis about the transfer of energy and that, as in the case of many of the earlier transfer hypotheses described in Ellison's talk, the effect of viscosity does not have a clear physical basis.

Another recent attempt to determine theoretically the form of the energy spectrum at wave-numbers large compared with the dissipation wave-number (E. A. Novikov, *Dok. Akad. Nauk. S.S.S.R.*, vol. 139, 1961, p. 331) was reported. The method was to use the known asymptotic behaviour of a single Fourier component of the velocity disturbance to a persistent uniform straining motion, in the manner of an earlier calculation of the spectrum of a convected scalar quantity (G. K. Batchelor, *J. Fluid Mech.*, vol. 5, 1959, p. 113). It appeared, after some discussion at the Colloquium, that the method cannot be used (at any rate, not without modification) for the energy spectrum since the total energy of the disturbance to a uniform straining motion is ultimately dominated by Fourier components which are slow to take up their asymptotic orientation; and the total disturbance energy actually tends ot infinity if the straining motion persists, as already shown by Pearson (*J. Fluid. Mech.*, vol. 5, 1959, p. 274). The way in which the method should be modified to take account of changes in the straining motion, and thereby to avoid the indefinite growth of the disturbance energy, remains to be found.

KRAICHNAN made the following remark about this work by NOVIKOV and the earlier related theory of TOWNSEND (*Proc. Roy. Soc.*, A, vol. 208, 1951, p. 534). Both of these investigations postulate an energy-transfer mechanism such that the energy-dynamics of this very high wave-number range is determined by the straining action of lower wave-numbers. The mechanism involves principally the interaction of the wave-numbers $\gg k_s \left[= \left(\frac{\varepsilon}{\nu^3}\right)^{1/4} \right]$ with wave-numbers of order k_s; the latter contain most of the vorticity and therefore exert most of the straining action. Thus they postulate a transfer process which depends on non-local interactions in wave-number space, in contrast to the local cascade process usually assumed in the inertial range.

Several years ago, he (KRAICHNAN) had investigated the energy-transfer in the far-dissipation range on the basis of the so-called direct-interaction approximation. The result (*J. Fluid Mech.*, vol. 5, 1959, p. 497) for the spectrum had the form

$$E(k) \propto k^3 \exp\left(-\frac{k}{k_c}\right), \tag{1}$$

where the wave-number k_c is a parameter determined by the theory. It appeared that the non-local interactions appealed to in TOWNSEND's theory played a negligible role in the energy transfer. Their contribution was completely overshadowed by that of interactions among triads of FOURIER modes all three of whose wave-numbers had the same order of magnitude. In other words, it was found that the local cascade process characteristic of the inertial range continued to be dominant in the far-dissipation range.

For sufficiently high k, the spectrum (1) is very much larger in magnitude than the very rapidly decreasing spectrum given by TOWSEND's theory. This is consistent with the conclusion that the interactions retained in the latter theory represent only a negligible part of the total energy-transfer for such k.

There is reason to believe, according to KRAICHNAN, that (1), in contrast to the inertial-range predictions of the direct-interaction approximation, is actually an asymptotically exact result for the far-dissipation range. The errors produced by the direct-interaction approximation in the inertial range may be very crudely described as due to an inaccurate treatment of the relaxation effects, upon triple correlations, of an effective dynamical viscosity. In the far-dissipation range, these errors appear to be negligible because of the dominance of actual viscosity over the effective dynamical viscosity.

If (1) is correct, the ineffectiveness of the non-local interactions retained in TOWNSEND's theory may be given a simple qualitative interpretation which is based on the non-linearity of the dynamical process. The elementary dynamical interactions in the FOURIER representation are among *triads* of modes. In order for there to be a transfer of energy into a mode $\mathbf{k}$ due to interaction with modes $\mathbf{k}'$ and $\mathbf{k}''$, *both* of the FOURIER amplitudes $\mathbf{u}(\mathbf{k}')$ and $\mathbf{u}(\mathbf{k}'')$ must be non-zero. In general, the rate of energy transfer depends on both $E(k')$ and $E(k'')$. Now suppose $k \gg k_s$ and $k' \sim k_s$. In order for the triad of modes to interact, it must be possible to form a triangle from their wave-numbers. Hence, $k'' \approx k$. The essential point now is that the spectrum falls off so rapidly in the far-dissipation range that $E(k'')$ is very small and suppresses strongly the energy-transfer associated with this non-local interaction.