SECTION

DIFFUSION ET POINT DE VUE DE LAGRANGE

DIFFUSION AND LAGRANGIAN EFFECTS

Président : Professeur S. CORRSIN

THE MATHEMATICAL NATURE OF THE PROBLEM OF RELATING LAGRANGIAN AND EULERIAN STATISTICAL FUNCTIONS IN TURBULENCE

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SOMMAIRE

A) Les dépendances et les relations entre les hypothèses d'homogénéité, isotropie, stationnarité et incompressibilité sont examinées du point de vue de leurs correspondances dans les représentations de Lagrange et d'Euler.

On trouve que, dans un fluide incompressible, l'isotropie et l'homogénéité dans la représentation d'Euler entraînent la propriété correspondante dans celle de Lagrange; que l'homogénéité eulérienne implique l'égalité des mesures des quantités eulériennes et lagrangiennes relatives à un point unique; et que l'homogénéité et la stationnarité eulérienne entraînent la stationnarité des fonctions statistiques lagrangiennes dépendant d'un seul point de l'espace.

Il est à noter que les fonctions lagrangiennes dépendant de plus d'un point de l'espace ne peuvent être stationnaires, et que les liaisons énumérées ci-dessus, ne sont pas nécessairement réciproques.

Il est d'autre part mentionné la nécessité de faire des hypothèses dynamiquemeent contradictoirs pour le domaine d'Euler, afin d'obtenir la stationnarité pour les fonctions statistiques lagrangiennes en un point. Le rôle important de l'hypothèse d'incompressibilité est discuté.

B) Le problème concernant l'obtention des statistiques de trajectoires de points est examiné.

Pour éclairer la question, nous décrirons les problèmes de l'obtention du champ des vitesses :

—suivant un chemin prescrit analytiquement (eulérien);

— suivant un chemin aléatoire, mais statistiquement indépendant;

— suivant un chemin statistiquement dépendant (lagrangien).

Une technique est décrite commençant par la formule de RICE-KAC, pour la probabilité d'annulation d'une fonction aléatoire, pour obtenir la probabilité fonctionnelle d'une trajectoire de point, durant un intervale de temps.

La simplification qui consiste à admettre l'incompressibilité est discutée.

L'attention est attirée sur le problème d'intégration de la probabilité fonctionnelle (dans l'espace des fonctions) de façon à donner les densités usuelles.

SUMMARY

A) The implications of, and relationships between, assumptions of homogeneity, isotropy, stationarity and incompressibility are examined for corresponding Lagrangian and Eulerian fields. It is found that in an incompressible fluid, isotropy and homogeneity of the Eulerian field imply the corresponding property of the Lagrangian field; that homogeneity of the Eulerian field implies equality of averages of Eulerian and Lagrangian quantities dependent on a single point; and that homogeneity and stationarity of the Eulerian field imply stationarity of Lagrangian statistical functions dependent on a single space point. It is noted that Lagrangian functions dependent on more than one space point cannot be stationary and that the implications given above do not necessarily work the other way. The necessity is mentioned of making dynamically inconsistent assumptions for the Eulerian field in order to obtain stationarity for Lagrangian one-point statistical functions. The strong role of the assumption of incompressibility is discussed.

B) The problem of obtaining the statistics of point trajectories is examined. For illustrative purposes, the problems are described of obtaining the velocity field : on an analytically prescribed path (Eulerian); on a random, but statistically independent path; on a statistically dependent path (Lagrangian). A technique is described, beginning from the RICE-KAC formula for the probability of occurrence of zeros of a random function, for obtaining the functional probability of a point trajectory during a time interval. The simplification of assuming incompressibility is discussed. Attention is given to the problem of integrating the functional probability (in function space) so as to give ordinary densities.

INTRODUCTION

Since the pioneering paper of G. I. TAYLOR in 1921 [1]* turbulent diffusion has been approached largely by expressing quantities of interest in terms of Lagrangian statistical functions about which simplifying assumptions are made with only limited reference to the Eulerian field. Generally speaking, with the exception of small time lag expansions [2], and the work on relative diffusion making use of the KOLMOGOROFF theory [3], the diffusion work has proceeded parallel to and independently of the work on the Eulerian field. Even the phenomenological descriptions which use the empirical evidence for a gaussian distribution of displacement of points tagged at a fixed initial location [4] leave open the question of the quantitative description of the diffusion coefficient. Some of the assumptions made in working with Lagrangian quantities are so natural as to appear obvious, and are sometimes made without the realization that they are assumptions (e. g. — the assumption that statistical functions at a material point are equal to the corresponding Eulerian functions — [5]). In some cases apparently impermissible extensions of the assumptions are unwittingly made (e. g. — that the two-point Lagrangian correlation tensor is stationary — [6]). Generally speaking, direct attacks on the general problems of relating Lagrangian and Eulerian quantities have been difficult to find.

In this paper we will try to give a general picture of the current state of knowledge about these problems. We have purposely omitted reference either to the various phenomenological descriptions, or to the descriptions which depend on the simplifications

* Numbers in square brackets refer to bibliography at end of paper.

arising from a particular dynamical theory. We wish to limit ourselves rather to the exact mathematical inferences which can be drawn relating the Eulerian and Lagrangian fields. We are going to split the discussion in a way which seems natural, into a discussion of the relationship of the various assumptions made regarding the Eulerian and Lagrangian fields, and a discussion of the general problem of relating Lagrangian and Eulerian statistical functions. Bridging the gap between the two sections is a theorem relating Lagrangian and Eulerian statistical functions depending on a single space point [7]. A technique used in the first section is that of formal series expansion, and this theorem is used to evaluate the coefficients.

1. — Homogeneity, isotropy, stationarity and incompressibility

Formally, the Lagrangian velocity field can be expanded in a power series in time :

$$\mathbf{v}(\mathbf{a}, t) = \mathbf{u}(\mathbf{r}(\mathbf{a}, t), t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial t^n} \mathbf{v}(\mathbf{a}, t) \Big|_{t=0} t^n \tag{1}$$

where

$$\mathbf{r}(\mathbf{a}, t) = \mathbf{a} + \int_0^t \mathbf{v}(\mathbf{a}, \tau)\, d\tau \tag{2}$$

and the coefficients are all Eulerian quantities :

$$\frac{\partial^n}{\partial t^n} \mathbf{v}(\mathbf{a}, t) \Big|_{t=0} = \left(\frac{\partial}{\partial t} + u_j(\mathbf{x}, t) \frac{\partial}{\partial x_j} \right)^n \mathbf{u}(\mathbf{x}, t) \Big|_{\substack{t=0 \\ \mathbf{x}=\mathbf{a}}} \tag{3}$$

Any statistical function of the Lagrangian field can therefore be represented formally as a power series in time with Eulerian coefficients. We can conclude immediately that

$$\text{Eulerian homogeneity} \Rightarrow \text{Lagrangian homogeneity} \tag{4}$$

$$\text{Eulerian isotropy} \Rightarrow \text{Lagrangian isotropy} \tag{5}$$

This technique could be applied formally in the reverse direction : that is, the Eulerian Field could be expressed as a formal power series in time, using the fact that the present Eulerian field is the Lagrangian field for some set of (time dependent) past locations :

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{v}(\mathbf{a}(\mathbf{x}, t)\, t) \tag{6}$$

However, the coefficients in such a series are not pure Lagrangian quantities, since they involve the strain tensor $\partial r_i(\mathbf{a}, t)/\partial a_j$ in Lagrangian co-ordinates, and consequently are difficult to interpret. Tentatively we can conclude that if statistical functions involving the Lagrangian field *and* the strain field are homogeneous, then the Eulerian field is homogenous, with a similar conclusion mutatis mutandis for isotropy. This technique, of course, cannot handle cases in which the Lagrangian field may become asymptotically homogeneous, such as fully developed turbulent pipe flow. It should be noted that so far we have not used the restriction of incompressibility, though we should probably require the flow to be everywhere subsonic.

In order to carry these relations further, it is necessary to use a theorem of Lumley [7] (similar to Liouville's theorem — see [8], p. 15). Let us imagine first a finite region R of turbulent fluid bounded by a solid surface on which the velocity vanishes; then, if we integrate any function of Lagrangian particle position [e. g. $u_1^n(\mathbf{r}(\mathbf{a}, t), t)$]

then, if we integrate any function of Lagrangian particle position [e. g. $u_1^n(\mathbf{r}(\mathbf{a}, t), t)$] over this volume, we can make a change to Eulerian variables, including the density ration as the Jacobian —

$$\int_R F(\mathbf{r}(\mathbf{a}, t), t)\, d\mathbf{a} = \int_R F(\mathbf{x}, t) \frac{\rho(\mathbf{x}, t)}{\rho(\mathbf{a}(\mathbf{x}, t), 0)}\, d\mathbf{x} \tag{7}$$

where $\mathbf{a}(\mathbf{x}, t)$ is the location at $t = 0$ of a point which at t finds itself at $\mathbf{x}$. For an incompressible fluid, the density ratio is unity, and taking ensemble averages we have

$$\int_R \overline{F(\mathbf{r}(\mathbf{a}, t), t)}\, d\mathbf{a} = \int_R \overline{F(\mathbf{x}, t)}\, d\mathbf{x} \tag{8}$$

If the Eulerian field were homogeneous (impossible because of the solid wall) then from what we have done above, the Lagrangian field would be also, and the two functions could be removed from the integrals —

$$\overline{F(\mathbf{r}(\mathbf{a}, t), t)} = \overline{F(\mathbf{x}, t)} \tag{9}$$

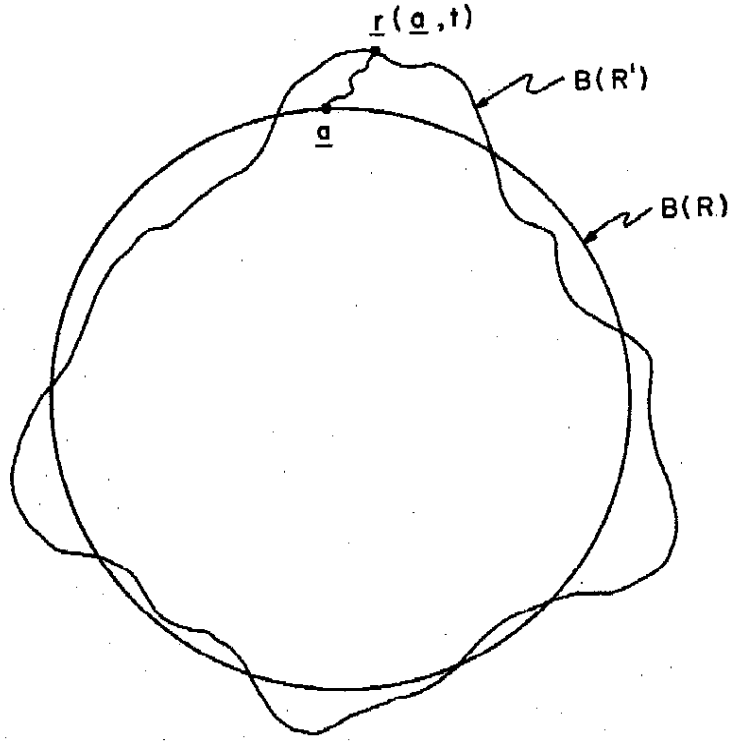

FIGURE 1
Motion of an initially spherical surface in turbulent fluid.

The unrealistic restriction to a solid boundary can easily be removed. An exact proof is not difficult, and is given with slightly greater generality in [7]; for our purposes, however, an heuristic argument will suffice. Suppose our space is filled with turbulent fluid in homogeneous motion. Let R be any region. Then the second integral, if the fluid is incompressible, will be over the same volume, but over a new region (say R′) since the boundaries will have moved (see Figure 1). We can estimate how

far they will have moved, however — if u' is the r. m. s. fluctuating velocity, then $u't$ is an estimate of the extent of the motion, and $u't\,S(R)$, where S(R) is the surface of the region, is an estimate of the volume by which the regions fail to overlap. If we divide by the volume V(R), and take the integrals both over the original region R the difference is estimated by

$$u't\,\frac{S(R)}{V(R)}\,\overline{F(\mathbf{x},t)} \tag{10}$$

This can be made as small as we like at fixed t by taking the region large enough. Instead of (8) we have

$$\lim_{V(R)\to\infty}\frac{1}{V(R)}\int_R \overline{F(\mathbf{r}(\mathbf{a},t),t)}\,d\mathbf{a} = \lim_{V(R)\to\infty}\frac{1}{V(R)}\int_R \overline{F(\mathbf{x},t)}\,d\mathbf{x} \tag{8a}$$

Again, Eulerian homogeneity (implying Lagrangian homogeneity) implies (9). Note that incompressibility seems to be essential, since otherwise we are left with a density ratio in the averages which is neither an Eulerian nor a Lagrangian quantity. If incompressibility is not assumed, and homogeneity permits removal of the averages from the space average, it is difficult even to show that $\overline{\mathbf{v}} = 0$ without assuming isotropy, since

$$\overline{\mathbf{v}} = \overline{\mathbf{u}\,\frac{\rho}{\rho_0}} \tag{11}$$

Using this theorem, if we let

$$F(\mathbf{x},t) = e^{i\lambda_j u_j(\mathbf{x},t)}$$

(the average of this will be just the characteristic function) we find that, for a homogeneous incompressible field, the entire distribution of Lagrangian velocity at any instant is identical with the Eulerian one. Note, however, that we cannot apply our theorem to an F that is dependent on more than a single Lagrangian point or time, so that the joint distributions at two points or times must differ. This same situation will arise again below, in the discussion of stationarity.

We can apply this theorem directly to the Lagrangian auto-correlation,

$$\overline{v_i(\mathbf{a},t)\,v_j(\mathbf{a},t+\tau)} = \overline{u_i(\mathbf{r}(\mathbf{a},t),t)\,u_j(\mathbf{r}(\mathbf{a},t+\tau),t+\tau)} \tag{12}$$

This can be expanded formally in a power series in τ —

$$\begin{aligned}\overline{v_i(\mathbf{a},t)\,v_j(\mathbf{a},t+\tau)} &= \overline{u_i(\mathbf{r}(\mathbf{a},t),t)\,u_j(\mathbf{r}(\mathbf{a},t),t)}\\ &+ \tau\,\overline{u_i(\mathbf{r}(\mathbf{a},t),t)\left\{u_{j,k}(\mathbf{x},t)\,u_k(\mathbf{x},t) + \frac{\partial}{\partial t}u_j(\mathbf{x},t)\right\}_{\mathbf{x}=\mathbf{r}(\mathbf{a},t)}}\\ &+ \cdots\end{aligned} \tag{13}$$

and all coefficient are functions only of a single Lagrangian point. Our theorem thus applies, and if we can assume Eulerian homogeneity, they can be replaced by the corresponding Eulerian averages

$$\begin{aligned}\overline{v_i(\mathbf{a},t)\,v_j(\mathbf{a},t+\tau)} &= \overline{u_i(\mathbf{x},t)\,u_j(\mathbf{x},t)}\\ &+ \tau\,\overline{u_i(\mathbf{x},t)\left\{u_{j,k}(\mathbf{x},t)\,u_k(\mathbf{x},t) + \frac{\partial}{\partial t}u_j(\mathbf{x},t)\right\}}\\ &+ \cdots\end{aligned} \tag{14}$$

Finally, the coefficients will be independent of time only if the Eulerian field is stationary. Thus, for the Lagrangian auto-correlation to be stationary, it seems as though we must assume that the Eulerian field is both stationary and homogeneous. There are two points to be noted : first, that this set of assumptions is, for an isotropic field, dynamically impossible — stationarity of the Lagrangian auto-correlation is therefore an approximation, and it is necessary to inquire, in a particular situation, to what extent the approximation is justified.

The second point is that even if the Lagrangian time-correlation could be regarded as stationary, it seems as though the Lagrangian space-time correlation could never be stationary. We cannot apply our theorem to this case, so we must reason rather heuristically. Consider the correlation in which the times are the same

$$\overline{v_i(\mathbf{a}, t)\, v_j(\mathbf{a}', t)} \tag{15}$$

Consider two material points whose initial separation is, say, of the order of the KOLMOGOROFF microscale. For small time, the correlation will be excellent; if $i = j$, we expect to be able to make it as close to unity as we please by taking $\mathbf{a}'$ close enough to $\mathbf{a}$. As t increases, however, the points must presumably wander apart, ultimately being separated by distances greater than the scale of the energy containing eddies. By waiting long enough we expect to find the correlations as close to zero as we please. Thus stationarity is not possible. The statement that the points must wander apart is, of course, an assumption, essentially the same as that made in the theory of line stretching, and seems to the author to carry implications with regard to ergodicity, since it precludes the possibility of the system ever returning arbitrarily close to its initial state. Such a return, of course, is associated with Birkhoff's ergodic theorem [8], which presupposes motion taking place in a bounded region of phase space, but is presumably not a necessity for ergodicity. Comparisons may be made with Polya's theorem [9] for the unbounded random walk, which states that in three or more dimensions, return to an initial state becomes uncertain. Placing reflecting barriers on one or more dimensions, so as to leave the number of unbounded dimensions smaller than three, again makes return certain, and seems to have correspondences with Birkhoff's theorem. Of course, relative displacement of a pair of random walks is again a random walk, while the relative displacement of material points in a fluid certainly is not, but if we may draw qualitative conclusions, this may be only another way in which turbulence must be three dimensional. Perhaps in two dimensions lines do not stretch continually. It seems to the author that this general area would be a most profitable one for further mathematical work.

2. — Qualitative description of the general problem

In order to grasp the difficulties inherent in the general problem, let us begin with the equations governing the Eulerian-Lagrangian transformation

$$\mathbf{r}(\mathbf{a}, t) = \mathbf{a} + \int_0^t \mathbf{v}(\mathbf{a}, \tau)\, d\tau = \mathbf{a} + \int_0^t \mathbf{u}(\mathbf{r}(\mathbf{a}, \tau), \tau)\, d\tau \tag{16}$$

where $\mathbf{u}(\mathbf{x}, t)$ is the Eulerian velocity field, regarding which we may assume that we have all statistical information, and $\mathbf{r}(\mathbf{a}, t)$ is displacement of a material point, initially at $\mathbf{a}$. Statistical information regarding $\mathbf{r}$ is the desired end product. The difficulty

lies in the fact that we need the statistics of $\mathbf{u}(\mathbf{r}, t)$ to obtain those of $\mathbf{r}$, and it is not clear how to obtain the statistics of $\mathbf{u}(\mathbf{r}, t)$ from those of $\mathbf{u}(\mathbf{x}, t)$. A graded series of problems may serve to clarify this point.

a) To obtain the statistics of $\mathbf{u}(\mathbf{z}(t), t)$ on $[0, T]$, where $\mathbf{z}(t)$ is a prescribed path. This is an Eulerian problem.

b) To obtain the statistics of $\mathbf{u}(\mathbf{z}(t), t)$ on $[0, T]$ where $\mathbf{z}(t)$ is a random path, with a prescribed distribution, but statistically independant of $\mathbf{u}$. This is no longer Eulerian, but is straightforward. Consider, for example

$$\text{Prob}\{\mathbf{u}(\mathbf{z}(t), t) \cong \mathbf{u}^*(t)\} =$$

$$\sum_{Z^*} \text{Prob}\{\mathbf{u}(\mathbf{z}(t), t) \cong \mathbf{u}^*(t) \mid \mathbf{z}(t) \cong \mathbf{z}^*(t)\} \text{Prob}\{\mathbf{z}(t) \cong \mathbf{z}^*(t)\} \quad \text{on} \quad [0, T] \qquad (17)$$

The two probabilities being independent by assumption, the first is an Eulerian quantity.

c) To obtain the statistics of $\mathbf{u}(\mathbf{r}(t), t)$ where

$$\mathbf{r}(t) = \int_0^t \mathbf{u}(\mathbf{r}(\tau), \tau)\, d\tau$$

This is the full Lagrangian problem. We can still write (17)

$$\text{Prob}\{\mathbf{u}(\mathbf{r}(t), t) \cong \mathbf{u}^*(t)\} =$$

$$\sum_{Z^*} \text{Prob}\{\mathbf{u}(\mathbf{r}(t), t) \cong \mathbf{u}^*(t) \mid \mathbf{r}(t) \cong \mathbf{z}^*(t)\} \text{Prob}\{\mathbf{r}(t) \cong \mathbf{z}^*(t)\} \quad \text{on} \quad [0, T] \qquad (18)$$

but the distributions are not now independent. In order for $\mathbf{r}(t)$ to have followed the path $\mathbf{z}^*(t)$, the velocity field must be a very particular one, so that each probability is strongly dependent on the other.

3. — A method of approach

One way of attacking this problem is to discretize the time axis, considering (16) then as a collection of simultaneous algebraic equations (See Figure 2)

$$0 = t_1 < t_2 < \ldots < t_n = T$$

$$\mathbf{r}(\mathbf{a}, t_i) = \mathbf{a} + \sum_{j=1}^{i} \mathbf{u}(\mathbf{r}(\mathbf{a}, t_j), t_j)\, \Delta_j \qquad (19)$$

where

$$\Delta_j = t_{j+1} - t_j$$

If we consider the set of functions

$$\mathbf{f}_i(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathbf{x}_i - \mathbf{a} - \sum_{j=1}^{i} \mathbf{u}(\mathbf{x}_j, t_j)\, \Delta_j \qquad i = 1, \ldots, n \qquad (20)$$

and ask for the probability that they have a zero in the vicinity of the points $\mathbf{x}_i$, this is just the probability that (16) has a solution passing in the vicinity of a test function $\mathbf{x}(t)$, (say the « string » function, with $\mathbf{x}(t_i) = \mathbf{x}_i$). This would be the functional probability of the solution to (16). Let's defer for a moment the question of how to utilize a functional probability, and see if we could obtain it.

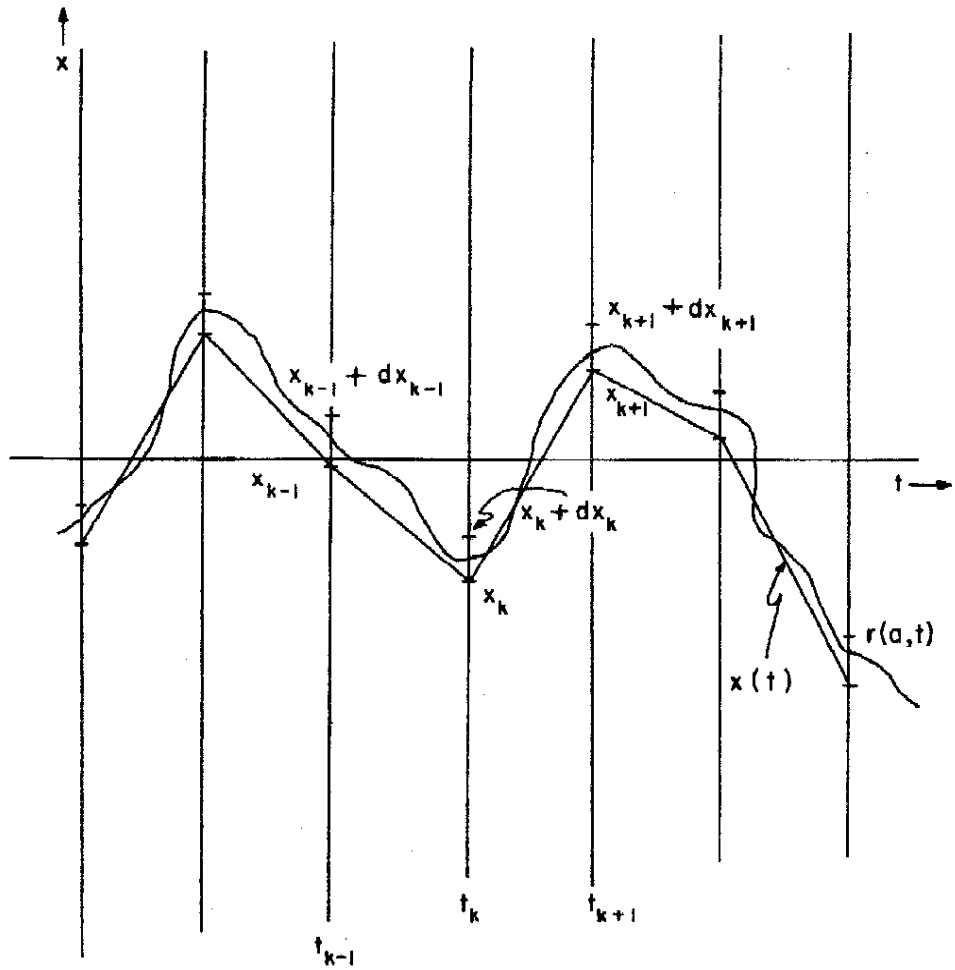

FIGURE 2

One dimensional example. A rather coarse division of the time axis with $x(t)$ and the dx_i shown, and a realization of $r(t)$ which happens to pass through all the « windows ».

We recall the RICE-KAC [10], [11] formula for the probability of occurrence of a zero of a random function $f(x)$ in the vicinity of x :

$$dx \int_{-\infty}^{+\infty} \mathrm{P}(0, v; x) \mid v \mid dv \tag{21}$$

where $\mathrm{P}(u, v; x)$ is the probability that f takes on at x a value within du of u, while the slope is within dv of v. This formula can be extended [7] to a set of vector functions of a set vectors, like (20). The form is similar

$$d\mathbf{x} \int_{-\infty}^{+\infty} \mathrm{P}(\mathbf{0}, \ldots, \mathbf{0}, \Pi_{11}, \ldots, \Pi_{nn}; \mathbf{x}_1, \ldots, \mathbf{x}_n) \mid \| \mathbf{H}_{ij} \| \mid d\mathbf{H}$$

where $\mathbf{H}_{ij}$ is an estimate for $\dfrac{\partial \mathbf{f}_i}{\partial \mathbf{x}_j}$ a 3×3 matrix, and $\| \mathbf{H}_{ij} \|$ is the determinant of

the matrix with n^2 blocks, each 3×3. This is somewhat formidable, since we must carry out $9\ n^2$ integrations. In an incompressible fluid, however, we have an enormous simplification, since it turns out that

$$\left\| \frac{\partial \mathbf{f}_t}{\partial \mathbf{x}_j} \right\| = \mathrm{e}^{-\int_0^T u_{i,i}(\mathbf{x}(t),\, t)\, dt} \equiv 1 \tag{23}$$

and the integration in (22) can be carried out immediately, giving

$$d\mathbf{x}\, \mathrm{R}(\mathbf{0}, ..., \mathbf{0}; \mathbf{x}_1, ..., \mathbf{x}_n) \qquad \text{(say} \tag{24}$$

the probability that

$$\mathbf{x}(t) - \mathbf{a} - \int_0^t \mathbf{u}(\mathbf{x}(\tau), \tau)\, d\tau \tag{25}$$

is within $d\mathbf{x}\ (t)$ of zero on [0, T]. Since we specify a path beforehand, $\mathbf{x}(t)$, and ask for the probability density that the equation is satisfied by it, this is Eulerian information.

4. — How to deal with the functional probability

Supposing that we have a functional probability density, $\mathrm{P}\{\mathbf{x}\overset{\mathrm{T}}{(t)}_0\}$, this is far more information than we want or need, for a physical problem. We would like instead, say, the probability density that $\mathbf{r}(\mathbf{a}, t) = \mathbf{x}$, and similar quantities. To obtain these, we must integrate in function space over all paths that end at the point of interest — e. g. —

$$\int_{\mathbf{x}(\mathrm{T}) = \mathbf{x}} \mathrm{P}\{\mathbf{x}\overset{\mathrm{T}}{(t)}_0\} d\Omega \tag{26}$$

At the present state of mathematics, such integrations can be carried out only under very special circumstances. One of these gives rise to the Wiener integrals [12] which correspond to $\mathrm{P}\{\mathbf{x}\overset{\mathrm{T}}{(t)}_0\}$ having the form appropriate to a particle undergoing Brownian motion : i. e. — Gaussian and separable. This, of course, is of no interest in turbulence. Even if we assume that the Eulerian field is a Gaussian process, $\mathrm{P}\{\mathbf{x}\overset{\mathrm{T}}{(t)}_0\}$ will not be Gaussian. The only other situation in which the integration can be carried out gives rise to the Feynman integrals [13]. These correspond to the non-relativistic motion of an electron. The integrand is an exponential, with exponent quadratic in the action. When this is expressed as the least action plus the departure from least action path, the part dependent on the end point (being an exponential with the square of the action along the least action trajectory for exponent) can be removed from the integral, the integrand then being independent of the end point. The integration need not be carried out, since the value of the integral is then a normalization constant.

Since we have no such set of fortuitous circumstances to aid us, we are momentarily at a loss to carry out the integration. In the meantime at least a machine integration could be performed, using the Monte-Carlo technique, and making for the Eulerian field the somewhat unrealistic assumption of a Gaussian process, with a homogeneous, stationary, isotropic, incompressible space-time correlation.

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THEORIES OF TURBULENT DISPERSION *

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SOMMAIRE

Une vue d'ensemble est donnée de travaux analytiques antérieurs classiques concernant le problème de la dispersion turbulente, compte non tenu des effets du mouvement moléculaire.

SUMMARY

A survey is made of some past and current analytical work on the problem of turbulent dispersion, ignoring the effects of molecular motion.

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* Supported by the National Science Foundation under Grant 6-4542 and the Air Force Office of Scientific Research under Cont. AF 49(638)248.

properties of the turbulence field. Especially for steady state problems, this includes such molecular analogies as the turbulent diffusivity,

$$\overline{\theta u_i} = \mathcal{D}_T \frac{\partial \bar{\Theta}}{\partial x_i}, \tag{6}$$

of which the outline may be attributed to BOUSSINESQ (1877), and the « mixing length » theories, which here would simply suggest that

$$\mathcal{D}_T = u' l, \tag{7}$$

with u' the r.m.s. turbulent velocity and l a length characteristic of the turbulent large structure. The mixing length concept was introduced by TAYLOR (1915). In later work [TAYLOR (1938)] he associated $\mathcal{D}_T$ with Lagrangian properties [see the following section], and pointed out that l, hence $\mathcal{D}_T$, must be a function of time for this problem of unsteady dispersion.

The representation of a transport rate as the product of a diffusivity and a mean gradient can be correct in principle only when the transport mechanism has a characteristic length much less than that of the $\bar{\Theta}$ field [see, for example, BOSWORTH (1952) Ch. I]. It follows that an expression like (6), even with $\mathcal{D}_T = fn.\ (t)$, is wrong in principle except for asymptotically large times [CORRSIN, 1957]. Then it turns out also that $\mathcal{D}_T \to$ constant.

Since we are not concerned with molecular diffusivity in this discussion, we can write (1) and (3) as

$$\frac{\partial \Theta}{\partial t} + u_i \frac{\partial \Theta}{\partial x_i} = 0, \tag{8}$$

or

$$\frac{\partial \Theta}{\partial t} + \frac{\partial}{\partial x_i}(u_i \Theta) = 0, \tag{8a}$$

and

$$\frac{\partial \bar{\Theta}}{\partial t} + \frac{\partial}{\partial x_i}(\overline{u_i \theta}) = 0. \tag{9}$$

By subtracting (9) from (8), multiplying by velocity and introducing the Navier-Stokes equations, we can deduce an equation starting with $\frac{\partial}{\partial t}(\overline{u_i \theta})$. But this contains a number of higher moments, so the system is still indeterminate — as will be any finite hierarchy of such equations. At some point a simplifying assumption must be introduced to truncate the hierarchy and render it determinate. (6) is perhaps the simplest example of such an assumption, and yields a « diffusion equation. »

Other assumptions have been applied in what seems to be a mixed Eulerian-Lagrangian formulation starting with (2) and (8) [ROBERTS (1960); KRAICHNAN (1961)]. In this connection, note that the indelibility of the tagging enables us to write down immediately the « solution » to (8) with (2) as initial condition

$$\Theta(\mathbf{x}, t) = \delta[\mathbf{x} - \mathbf{X}(o, t)] \tag{10}$$

with average concentration

$$\bar{\Theta}(\mathbf{x}, t) = \overline{\delta[\mathbf{x} - \mathbf{X}(o, t)]} \tag{11}$$

$\mathbf{X}(\mathbf{a}, t)$ is the *(Lagrangian)* particle position field. Thus the desired statistical solution

to our stated problem, $\bar{\Theta}(\mathbf{x}, t)$, coincides with the probability density function of $\mathbf{X}(o, t)$. If this turbulence property is given, the problem is trivially solved. Generally it is *Eulerian* properties which are accessible, so (11) simply says that the dispersion problem is most directly viewed as Lagrangian.

(10) and (11) can be written in terms of Eulerian fields by expanding $\mathbf{X}$ in time :

$$\mathbf{X}(o, t) = \mathbf{v}\ (o, o)\ t + \mathbf{v}_t(o, o)\ \frac{t^2}{2\,!} + \ldots\ldots, \tag{12}$$

where $\mathbf{v}\ (\mathbf{a}, t)$ is the Lagrangian velocity field, schematically connected to $\mathbf{u}(\mathbf{x}, t)$ by

$$\mathbf{v}(\mathbf{a}, t) = \mathbf{u}\,[\mathbf{X}(\mathbf{a}, t), t] \tag{13}$$

so that

$$\frac{\partial}{\partial t}\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}$$

Thus (12) can be expressed in Eulerian terms :

$$\mathbf{X}(0, t) = \mathbf{u}\,(0,0)\ t + \left\{\mathbf{u}_t\,(0, 0) + [\mathbf{u}\cdot\nabla\mathbf{u}]_{0,0}\right\}\ \frac{t^2}{2\,!} + \ldots\ldots, \tag{15}$$

to be substituted into (11). In any case, it appears that this « solution » is inherently Lagrangian. This procedure is used by Roberts (1960) ; his distinctions between Eulerian and Lagrangian fields are sometimes not emphasized.

For *vanishingly small times* the non-infiniteness of acceleration constrains the fluid point to travel in a linear extension of $\mathbf{u}\ (0,0)$. In this limit (15) reduces to

$$\mathrm{X} \to \mathbf{u}\,(0,0)\ t, \tag{16}$$

(11) to

$$\bar{\Theta}(\mathbf{x}, t) \to \overline{\delta\,[\mathbf{x} - \mathbf{u}\,(0,0)\ t]}, \tag{17}$$

i. e.

$$\bar{\Theta}(\mathbf{x}, t) \sim t \times p.d.f.\ \{\mathbf{u}\,(0,0)\}. \tag{18}$$

p.d.f. $\{\mathbf{u}\,(0,0)\}$ is the joint probability density function of the velocity components at $\mathbf{x} = 0$ and $t = 0$. With stationary, homogeneous $\mathbf{u}$,

$$\text{p.d.f. } \{\mathbf{u}\,(0,0)\} = \mathrm{P}_u\,(\mathbf{u}), \tag{19}$$

the fixed probability density of u_1, u_2, u_3. Since P_u is independent of time, it is clear that $\bar{\Theta}$ spreads linearly at small time.

As a final question in this section on Eulerian formulation, we may ask about the possible use of the Eulerian displacement field, $\xi\ (\mathbf{x}, t)$, a concept rarely used in mechanics. The meaning of the Lagrangian position field, $\mathbf{X}(\mathbf{a}, t)$, with initial position coordinate $\mathbf{a}$ defined by

$$\mathbf{a} \equiv \mathbf{X}\,(\mathbf{a}, 0), \tag{20}$$

is intuitively obvious. The connection between $\xi\ (\mathbf{x}, t)$ and $\mathbf{X}(\mathbf{a}, t)$ may be taken as

$$\xi\,(\mathbf{x}, t) = \mathbf{X}\,[\mathbf{a}\,(\mathbf{x}, t), t] - \mathbf{a}\,(\mathbf{x}, t) = \underline{\mathbf{x}} - \underline{\mathbf{a}}\,(\underline{\mathbf{x}}, t) \tag{21}*$$

where $\mathbf{a}\,(\mathbf{X}, t)$ is the inverse of $\mathbf{X} = \mathbf{X}\,(\mathbf{a}, t)$. The inverse of (21) is

$$\mathbf{X}\,(\mathbf{a}, t) = \mathbf{a} + \xi\,[\mathbf{X}\,(\mathbf{a}, t), t]. \tag{22}$$

From (21) we see that $\xi\ (\mathbf{x}, o) = 0$ by definition.

* I should like to thank J. L. Ericksen for a helpful discussion on this formulation.

Evidently $\xi(\mathbf{x}, t)$ is the net vector displacement which was traveled by the fluid particle found at time t at coordinate station $\mathbf{x}$. ξ was used, without being properly identified, by CORRSIN (1952) in a Lagrange-Euler formulation of the problem of temperature fluctuations in a linear temperature gradient across an isotropic turbulence.

In principle we can write the Navier-Stokes equations for $\xi(\mathbf{x}, t)$, and deduce equations for its correlation tensor and power spectrum. Even without the viscous terms, these equations are quite complex.

C. — Lagrangian Kinematic Formulation

The Eulerian approach was doubtless appreciated soon after the corresponding momentum analysis of REYNOLDS (1895)*, but the first paper actually presenting equation (3) seems to be one by KAMPÉ DE FÉRIET in 1937.

For the dispersion problem considered here, the Lagrangian approach, introduced by TAYLOR (1921), seems quite natural. With indelible tagging of a fluid point, e. g. at the origin at $t = 0$, it is obvious that the mean (Eulerian) concentration field at time t is precisely the probability density function of the random Lagrangian particle position $\mathbf{X}(o, t)$.

In general,

$$\mathbf{X}(\mathbf{a}, t) = \mathbf{a} + \int_0^t \mathbf{v}(\mathbf{a}, t_1)\, dt_1. \tag{23}$$

Choosing $\mathbf{a} = 0$ for simplicity,

$$\mathbf{X}(o, t) = \int_0^t \mathbf{v}(o, t_1)\, dt_1. \tag{24}$$

$\mathbf{v}(\mathbf{a}, t)$ is related to $\mathbf{u}(\mathbf{x}, t)$ by (13), so (23) can be written as an integral equation,

$$\mathbf{X}(\mathbf{a}, t) = \mathbf{a} + \int_0^t \mathbf{u}[\mathbf{X}(\mathbf{a}, t_1), t_1]\, dt_1. \tag{25}$$

Even if we give the simple probability density function of $\mathbf{v}$, we cannot predict that of $\mathbf{X}$ because of the integration. But the primary measure of dispersion is the mean square displacement, and TAYLOR deduced from (24) its connection with Lagrangian autocorrelation, e. g. $\mu_{ik}(o, \tau) \equiv \overline{v_i(\mathbf{a}, t)\, v_k(\mathbf{a}, t + \tau)}$:

$$\frac{1}{2}\frac{d}{dt}\overline{X_1^2} = \int_0^t \mu_{11}(o, \tau)\, d\tau. \tag{26}$$

In Brownian motion theory the corresponding quantity is the diffusivity, and TAYLOR (1935) pointed out that one might want to use this integral as a (time-variable) diffusivity in an equation like (6), even though (6) might be wrong in principle.

KAMPÉ DE FÉRIET (1939) showed that the integral of (26) can be written as

$$\overline{X_1^2}(t) = \int_0^t (t - \tau)\, \mu_{11}(o, \tau)\, d\tau. \tag{27}$$

Both he and BATCHELOR (1949) presented a tensorial generalization, and the complementary Fourier viewpoint with Lagrangian spectrum.

* See, for example, TAYLOR (1915).

TAYLOR was able to infer some asymptotic behaviours of $\overline{X_1^2}(t)$ directly from (26) and the necessary properties of $\mu_{11}(o,\tau)$. Since $\mu_{11} \to \overline{v_1^2}$ for $t \to 0$,

$$\overline{X_1^2} \to \overline{v_1^2}\, t^2, \tag{28}$$

in agreement with (18).

In the other limit he needed only to assume μ_{11} is integrable in order to arrive, for t large enough, at

$$\overline{X_1^2} \to 2\,\overline{v_1^2}\, T_{11}\, t, \tag{29}$$

where

$$T_{11} \equiv \frac{1}{\overline{v_1^2}} \int_0^\infty \mu_{11}(o,\tau)\, d\tau$$

is the Lagrangian integral (time) scale. He pointed out [TAYLOR (1935) (1)] that $\sqrt{\overline{v_1^2}}\, T_{11}$ is a plausible length to put into a mixing length theory, if you use one.

(29) shows just the behaviour familiar from Brownian motion, in which the easily observable times are all much larger than the molecular motion correlation time. This parabolic growth of $\sqrt{\overline{X_1^2}}$ together with the observed Gaussian shape for $\overline{\Theta}$ (rationalizable via a « central limit theorem »), corresponds to the hypothetical case of $\overline{\Theta}(\mathbf{x}, t)$ described by a classical diffusion equation. Of course, this does not prove the asymptotic validity of the diffusivity concept, but lends it support.

His work also indicated an extension of (28) by the introduction of the expansion in τ of μ_{11} :

$$\mu_{11}(o,\tau) \equiv \overline{v_1(\mathbf{a},t)\, v_1(\mathbf{a},t+\tau)} = \overline{v_1^2} \left\{ 1 - \frac{\tau^2}{\alpha_{11}^2} + \dots \right\}, \tag{30}$$

where

$$\alpha_{11}^2 \equiv -\,2\,\frac{\overline{v_1^2}}{\ddot{\mu}_{11}(o,o)} = 2\,\frac{\overline{v_1^2}}{\overline{\left(\frac{\partial v_1}{\partial t}\right)^2}} \tag{31}$$

α_{11} is the Lagrangian « time microsale », related to the mean square fluid particle acceleration, $\overline{\left(\frac{\partial v_1}{\partial t}\right)^2}$, as indicated. With (30), the small t form is generalized to

$$\overline{X_1^2}(t) \doteq \overline{v_1^2}\, t^2 \left(1 - \frac{1}{6}\,\frac{t^2}{\alpha_{11}^2}\right), \tag{32}$$

and TAYLOR (1935) gave a rough estimate for α_{11} in terms of the Eulerian « microscale » in space. For non-decaying turbulence this was improved by HEISENBERG (1948), whose prediction was in moderate agreement with experiments in decaying, grid-generated turbulence [UBEROI and CORRSIN (1953)]. See equation (57).

Equation (32) plus an estimate for α_{11} in terms of Eulerian properties constitute the first step in a formal procedure for expressing the (Lagrangian) measure of dispersion $\overline{X_1^2}(t)$ in terms of Eulerian moments, via the power series expansion of μ_{11} as in (30) :

$$\mu_{11}(o,\tau) = \overline{v_1^2} \left\{ 1 - \overline{v_{1_t}^2}\,\frac{\tau^2}{2!} + \overline{v_{1_{tt}}^2}\,\frac{\tau^4}{4!} - \dots \right\}. \tag{33}^*$$

* TAYLOR (1921).

For constant density fluid the one-point Lagrangian and Eulerian averages are equal [LUMLEY (1957)], so $\overline{v_1^2} = \overline{u_1^2}$. Each Lagrangian time derivative can be expressed in Eulerian terms, e. g.

$$\overline{\left(\frac{\partial v_1}{\partial t}\right)^2} = \overline{\left(\frac{\partial u_1}{\partial t} + u_k \frac{\partial u_1}{\partial x_k}\right)^2}, \tag{34}$$

$$\overline{\left(\frac{\partial^2 v_1}{\partial t^2}\right)^2} = \overline{\left[\left(\frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j}\right)\left(\frac{\partial u_1}{\partial t} + u_k \frac{\partial u_1}{\partial x_k}\right)\right]^2}, \tag{35}$$

etc*. Of course the acceleration can be replaced by the forces, so that

$$\overline{\left(\frac{\partial v_1}{\partial t}\right)^2} = \overline{\left(\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \nu \nabla^2 u_1\right)^2}, \tag{36}$$

as was done by TAYLOR (1935) and HEISENBERG (1948). In either case the higher power terms become complex indeed.

It is, of course, possible to extend Taylor's formulation (26), to the higher moments of $\mathbf{X}$ by repeated multiplication of (24). For example,

$$\overline{X_1^4}\ (t) = 8 \int_0^t \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \overline{v_1(t_1)\, v_1(t_2)\, v_1(t_3)\, v_1(t_4)}\, dt_4\, dt_3\, dt_2\, dt_1. \tag{37}$$

Writing $\tau' s$ for time differences, the integrand is

$$\overline{v_1(\mathbf{a}, t_1)\, v_1(\mathbf{a}, t_2)\, v_1(\mathbf{a}, t_3)\, v_1(\mathbf{a}, t_4)} = \mu_{1111}(o; \tau, \tau_1, \tau_2). \tag{38}$$

One may then be tempted to approximate these high order velocity moments in terms of μ_{11} by discarding the cumulants (quasi-normality). Such a step does not appear worthwhile here because the $\mathbf{X}$ -moments themselves are more likely to have negligible cumulants.

The spectral representation of (26) reveals another side of the coin :

$$\mathcal{D}_T(t) = \frac{1}{2}\frac{d}{dt}\overline{X_1^2} = \int_0^\infty \Phi_{11}(o, \omega)\, \frac{\sin(\omega t)}{\omega}\, d\omega, \tag{39}$$

where Φ_{11} is the Lagrangian frequency spectrum, the Fourier cosine transform of $\mu_{11}(o, \tau)$,

$$\Phi_{11}(o, \omega) = \frac{1}{\pi}\int_0^\infty \mu_{11}(o, \tau) \cos(\omega \tau)\, d\tau.$$

We see that

$$\mathcal{D}_T(o) = t \int_0^\infty \Phi_{11}\, d\omega) = t\, \overline{v_1^2},$$

and

$$\frac{\mathcal{D}_T(t)}{\mathcal{D}_T(o)} = \frac{1}{\overline{v_1^2}} \int_0^\infty \Phi_{11}(o, \omega)\, \frac{\sin(\omega t)}{\omega t}\, d\omega. \tag{40}$$

Figure (1) is a qualitative sketch of the two factors in the integrand of (40). As time progresses, $\Phi_{11}(o, \omega)$ remains fixed, but the « diffusivity filter function » continuously contracts toward zero frequency. Since they change continuously with time, the precise locations of zeroes at any instant are not so important; the envelope behaviour is of primary relevance.

* UBEROI (1954).

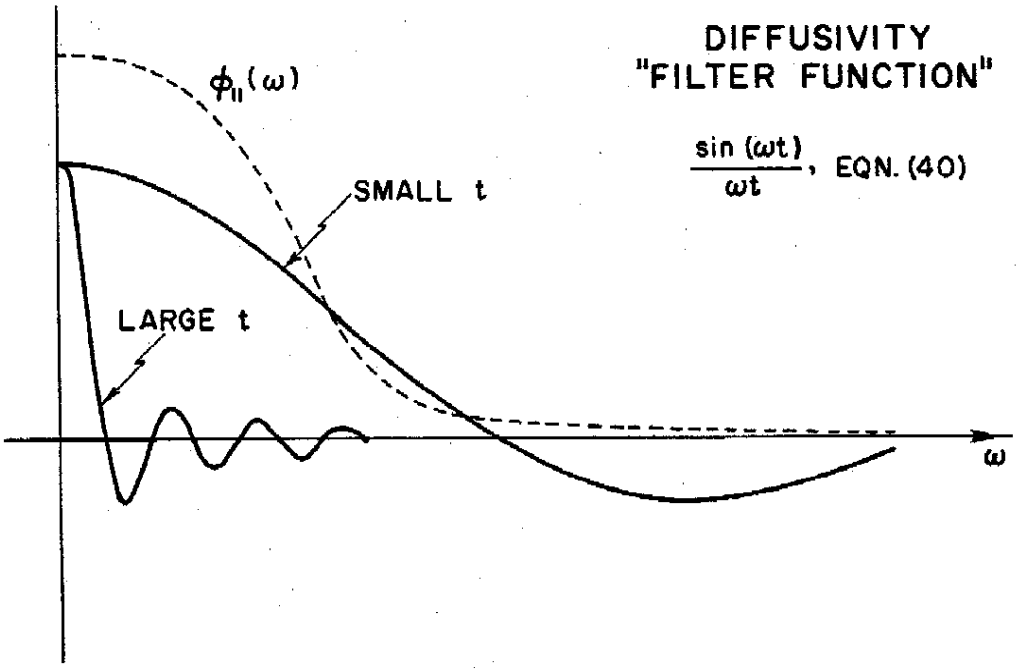

FIGURE 1

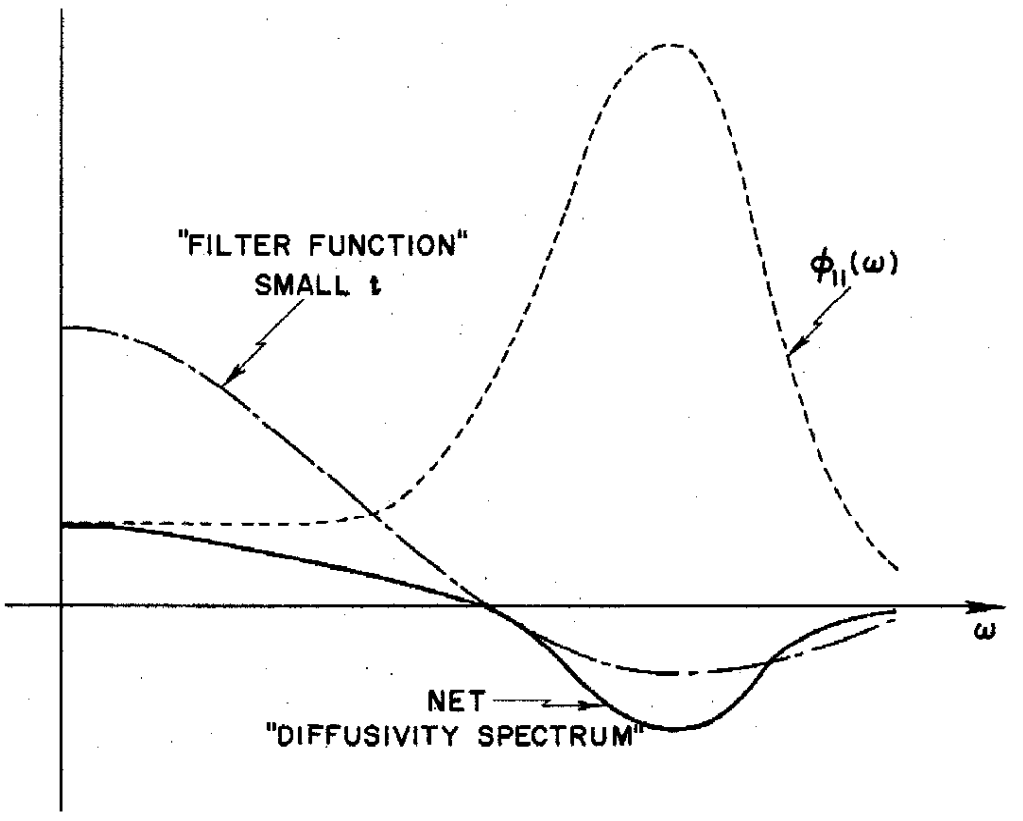

FIGURE 2

(40) presents the explicit measure of how much more efficient the low end of the spectrum is for dispersion. It must be emphasized, however, that if Φ_{11} happens to have a large enough maximum at high frequency, this (small eddy ?) region may dominate the dispersion for quite a while [figure (2)].

D. — Lagrangian Dynamic Formulation

In section C we identified the dispersion with Lagrangian displacement, suggesting that « the problem » could then be exemplified by the attempt to express $\overline{X_1^2}(t)$ in terms of the (experimentally and theoretically more accessible) Eulerian statistical functions, presumably by way of the Lagrangian velocity autocorrelation.

Of course fluid mechanics can, in principle, be worked entirely in the Lagrangian frame. One possible Lagrangian approach involves starting with the dynamic equations for $\mathbf{X}(\mathbf{a}, t)$, seeking then some equations for suitable moments. Following Gerber (1949), we can write one component as

$$\frac{\partial^2 X_1}{\partial t^2} = -\frac{1}{\rho} J\left(\frac{P, X_2, X_3}{a_1, a_2, a_3}\right) + \nu J\left\{\frac{J\left[\frac{\partial X_1}{\partial t}, X_2, X_3\right]}{a_1, a_2, a_3}, X_2, X_3\right\}$$
$$+ \nu J\left\{\frac{X_1, J\left[\frac{X_1, \frac{\partial X_1}{\partial t}, X_3}{a_1, a_2, a_3}\right]}{a_1, a_2, a_3}, X_3\right\}$$
$$+ \nu J\left\{\frac{X_1, X_2, J\left[\frac{X_1, X_2, \frac{\partial X_1}{\partial t}}{a_1, a_2, a_3}\right]}{a_1, a_2, a_3}\right\} \qquad (41)$$

J is the Jacobian, P is the static pressure field in Lagrangian coordinates. Even neglecting viscous forces, (41) and the other two components must yield awkward moment equations. A possible advantage for theory is that cumulant discards (quasi-normal hypotheses) may be a better approximation for displacement moments than for velocity moments. This is to be expected by « central limit reasoning ».

There is no obvious advantage to working in terms of Lagrangian velocity dynamics, since the dispersion problem is concerned with displacement.

E. — Lagrangian Diffusion Equation

Contaminant concentration rather than particle displacement can be chosen as dependent variable in a Lagrangian approach. Then the starting point would be the diffusion equation [in Eulerian frame, equation (1)]. Writing the concentration field in Lagrangian frame as

$$\Gamma(\mathbf{a}, t) = \Theta[\mathbf{X}(\mathbf{a}, t), t],$$

w can transform (1) into

$$\frac{\partial \Gamma}{\partial t} = \mathcal{D}\,\mathrm{J}\left\{\frac{\mathrm{J}\left(\dfrac{\Gamma, \mathrm{X}_2, \mathrm{X}_3}{a_1, a_2, a_3}\right), \mathrm{X}_2, \mathrm{X}_3}{a_1, a_2, a_3}\right\} + \mathcal{D}\,\mathrm{J}\left\{\frac{\mathrm{X}_1, \mathrm{J}\left(\dfrac{\mathrm{X}_1, \Gamma, \mathrm{X}_3}{a_1, a_2, a_3}\right), \mathrm{X}_3}{a_1, a_2, a_3}\right\} + \mathcal{D}\,\mathrm{J}\left\{\frac{\mathrm{X}_1, \mathrm{X}_2, \mathrm{J}\left(\dfrac{\mathrm{X}_1, \mathrm{X}_2, \Gamma}{a_1, a_2, a_3}\right)}{a_1, a_2, a_3}\right\} \tag{42}$$

If we wish to use (42) as our starting point, it is reasonable to take **X** as statistically given, just as we would wish to take **u** as prescribed if (1) were our starting point.

The initial condition for dispersion from a point source at the origin is

$$\Gamma(\mathbf{a}, o) = \delta(\mathbf{a}) \tag{43}$$

Obviously the diffusion terms in (42) will introduce some difficult moments into the equation for the Γ-moments.

Neglecting diffusion [corresponding to the Eulerian equation (8)], (42) reduces to

$$\frac{\partial \Gamma}{\partial t} = 0 \tag{44}$$

and the solution for our case, like (10), is

$$\Gamma(\mathbf{a}, t) = \delta(\mathbf{a}), \tag{45}$$

a trivial result.

The most likely future application of (42) seems to be in connection with estimation of the additional effect of molecular diffusion during turbulent diffusion, the problem discussed by Professor SAFFMAN at this Symposium.

F. — Kolmogorov Theory Applied to Lagrangian Correlation and Spectrum

Equation (9) for Eulerian transport, does not submit directly to application of the KOLMOGOROV (1941)* concepts of local isotropy and inertial range. In order to employ his approach we must find a quantity whose behaviour is dominated by the small eddies of the turbulence.

We can presumably apply KOLMOGOROV's concepts to the Lagrangian correlation or spectral function at large enough REYNOLDS number. Having done that, we can look for dispersion functions dominated by the correlation or spectrum region so evaluated.

For correlation behaviour we look at

$$\overline{[v_1(t+\tau) - v_1(t)]^2} = 2\overline{v_1^2} - 2\mu_{11}(\tau). \tag{46}$$

For small enough τ this is like acceleration, and may depend primarily on small

* See also BATCHELOR (1947) and BATCHELOR (1953).

structure, hence on kinematic viscosity ν and on dissipation rate ε. Possibly the restriction

$$\tau \ll \frac{1}{\overline{v_1^2}} \int_0^\infty \mu_{11}\, d\tau = T_{11} \tag{47}$$

will suffice. For dimensional reasons, we write

$$\overline{v_1^2} - \mu_{11}(\tau) = A \sqrt{\nu\varepsilon}\, \beta\left(\sqrt{\frac{\varepsilon}{\nu}}\,\tau\right). \tag{48}$$

β is presumably a universal function, A a constant. The limiting behaviour of β for $\tau \to o$ follows directly from a comparison of (48) and (30) :

$$\beta(s) \to s^2, \quad \text{for } s \to o. \tag{49}$$

If we narrow the range of τ further by a lower bound to keep it well above the viscous range, i. e. if

$$\sqrt{\frac{\nu}{\varepsilon}} \ll \tau \ll T_{11}, \tag{50}$$

it is expected that ν will vanish from (48). Therefore, in *this inertial subrange,*

$$\beta(s) \sim s, \tag{51}$$

and hence

$$\overline{v_1^2} - \mu_{11}(\tau) = C\varepsilon\tau. \tag{52}*$$

From (30) and (52), we can make a qualitative sketch of $\mu_{11}(\tau)$, figure 3.

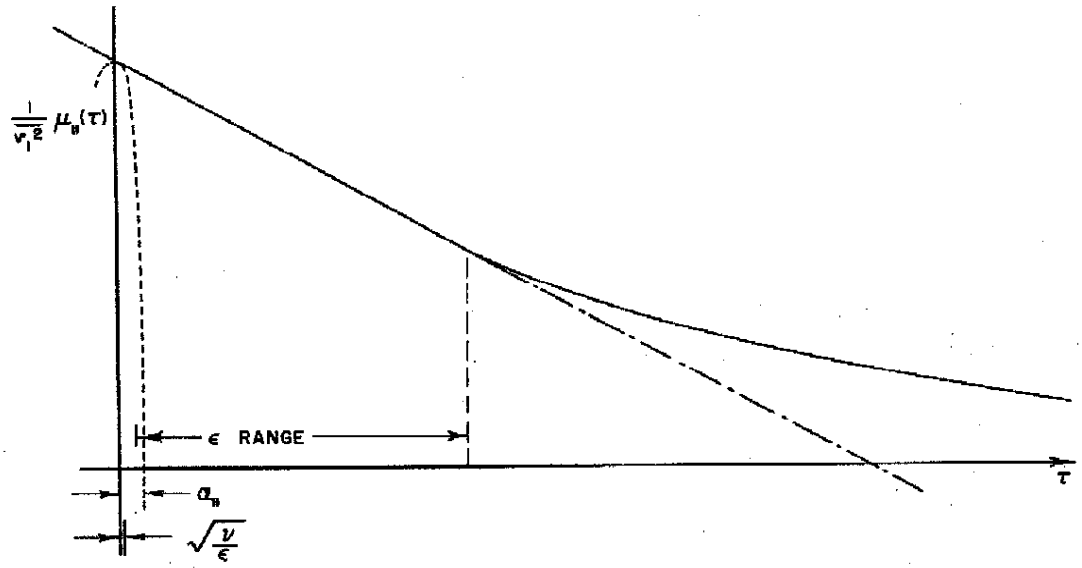

FIGURE 3

The corresponding spectral analysis gives, for $\omega \gg T_{11}^{-1}$,

$$\Phi_{11}(\omega) = B\nu\gamma\left(\sqrt{\frac{\nu}{\varepsilon}}\,\omega\right). \tag{53}$$

* Due to INOUE (1950).

And for the inertial range,

$$\frac{1}{T_{11}} \ll \omega \ll \sqrt{\frac{\varepsilon}{\nu}}, \tag{54}$$

we choose γ to eliminate ν from (53) :

$$\Phi_{11}(\omega) = B\varepsilon\omega^{-2}. \tag{55}$$

A rough estimate of the dimensionless constant C can be made by matching (52) to (30) at the KOLMOGOROV time $\sqrt{\frac{\nu}{\varepsilon}}$:

$$C \approx \frac{\overline{v_1^2}\ \nu^{1/2}}{\alpha_{11}^2\ \varepsilon^{3/2}}. \tag{56}$$

We recall [equation (32] that $2\frac{\overline{v_1^2}}{\alpha_{11}^2}$ is just the mean square acceleration. And $\sqrt{\frac{\nu}{\varepsilon}}$ is the KOLMOGOROV time, half the root-mean-square strain rate of the turbulence.

C can be estimated numerically if we introduce what amounts to HEISENBERG's (1948) estimate for $\frac{\lambda^2}{\overline{v_1^2}\ \alpha_{11}^2}$. Let u_1 be an Eulerian component, λ is the TAYLOR « microscale ». $\overline{u_1^2} = \overline{v_1^2}$ for this turbulence [LUMLEY (1957)]. For large $R_\lambda \equiv \frac{\sqrt{\overline{u_1^2}}\ \lambda}{\nu}$,

$$\frac{\lambda^2}{\overline{v_1^2}\ \alpha_{11}^2} \approx \frac{29}{R_\lambda}. \tag{57}^*$$

Using this to replace α_{11}^2 in (56), and replacing ε by the TAYLOR (1935) formula,

$$\varepsilon = 15\,\nu\frac{\overline{u_1^2}}{\lambda^2}, \tag{58}$$

we find

$$C \approx \frac{1}{2}, \tag{59}$$

so (52) becomes

$$\mu_{11}(\tau) \approx \overline{v_1^2} - \frac{1}{2}\varepsilon\tau. \tag{60}$$

A rough estimate of the constant B in (55) can be gotten from the inverse of (40) :

$$\mu_{11}(\tau) = \int_0^\infty \Phi_{11}(\omega)\cos(\omega\tau)\,d\omega. \tag{61}$$

Differentiating twice and letting $\tau = 0$,

$$-\ddot{\mu}_{11}(o) = \int_0^\infty \omega^2\,\Phi_{11}(\omega)\,d\omega. \tag{62}$$

Into the integral we put (55), and on the left we put (31) :

$$2\frac{\overline{v_1^2}}{\alpha_{11}^2} = \int_0^\infty \omega^2\,\Phi_{11}\,d\omega \approx B\varepsilon\int_{\frac{1}{T_{11}}}^{\sqrt{\frac{\varepsilon}{\nu}}} d\omega. \tag{63}$$

* See equation (34) of UBEROI and CORRSIN (1953).

The result is

$$B \approx 2 \frac{\overline{v_1^2}\, \nu^{1/2}}{\alpha_{11}^2\, \varepsilon^{3/2}}. \tag{64}$$

Introducing (58) and (57) again, we get the numerical estimate

$$B \approx 1 \tag{65}$$

A rough estimate of the ratio $\frac{\alpha_{11}}{T_{11}}$ can be gotten by

$$\overline{v_1^2} = \int_0^\infty \Phi_{11}\, d\omega \approx \varepsilon \int_{\frac{1}{T_{11}}}^{\sqrt{\frac{\varepsilon}{\nu}}} \frac{d\omega}{\omega^2}. \tag{66}$$

This gives

$$\overline{v_1^2} \approx \varepsilon\, T_{11}. \tag{67}$$

With (58) and (57), we can obtain

$$\frac{\alpha_{11}}{T_{11}} \approx \frac{3}{\sqrt{R_\lambda}}. \tag{68}$$

G. — Applicability to Single Particle Dispersion

As pointed out by Batchelor (1950), and displayed by his spectral expression (39) the simplest function of interest in dispersion, $\overline{X_1^2}\ (t)$ is probably dominated by the large (low frequency) eddies for all values of t. For $t \to 0$, (39) reduces to

$$\overline{X_1^2}\ (t) = t^2 \int_0^\infty \Phi_{11}\, (\omega)\, d\omega, \tag{69}$$

which is determined by the « energy-bearing » part of the spectrum. For increasing t, the « diffusivity filter function » in (40) discriminates even more against the large ω region (figure 1).

Consider, however, the « small t » series expansion of cos (ωt) in the time integral of (39) :

$$\overline{X_1^2}\ (t) = 2 \int_0^\infty \frac{\Phi_{11}}{\omega^2} \left\{ \omega^2 t^2 - \frac{\omega^4 t^4}{12} + \dots \right\} d\omega. \tag{70}$$

For very small $t \neq 0$,

$$\overline{X_1^2}\ (t) - \overline{v_1^2}\ t^2 \approx -\frac{t^4}{6} \int_0^\infty \omega^2\, \Phi_{11}\, d\omega, \tag{71}$$

a quantity whose « filter function » favors the higher frequencies. From (32) and (31) we note that the integral in (71) is the mean square particle acceleration.

If there is a time range for which (55) and (71) are both valid,

$$\overline{X_1^2}\ (t) \approx \overline{v_1^2}\ t^2 \left\{ 1 - \frac{1}{24} \frac{\varepsilon^{3/2}}{\overline{v_1^2}\, \nu^{1/2}}\, t^2 \right\} \tag{72}$$

Introducing the dimensionless time variable $\sqrt{\frac{\varepsilon}{\nu}}\, t$ (which must be $\gg 1$ for this case), and eliminating the residual ε with (58),

$$\overline{X_1^2}\,(t) \approx \overline{v_1^2}\,t^2\left\{1-\frac{1}{6\,R_\lambda}\left(\frac{\varepsilon}{\nu}\,t^2\right)\right\}. \tag{73}$$

For large enough R_λ, this truncation of (70) will be good even though $\sqrt{\frac{\varepsilon}{\nu}}\,t \gg 1$.

Casting about for other dispersion functions which may be dominated by eddies in the intertial range, we note that higher power terms in (70) will weight the high frequency end of even more.

A quantity which can be dependent on only a limited time range of the correlation is, for example,

$$\left(\frac{d\overline{X_1^2}}{dt}\right)_{t=t} - \left(\frac{d\overline{X_1^2}}{dt}\right)_{t=t'} = 2\int_{t'}^{t} \mu_{11}\,(o,\tau)\,d\tau. \tag{74}$$

When both t and t' satisfy (50), we should be able to approximate μ_{11} by (52).

On the other hand,

$$\overline{X_1^2}\,(t) - \overline{X_1^2}\,(t') = 2\int_{t'}^{t}\left\{\int_0^{t_1} \mu_{11}\,d\tau\right\} dt_1, \tag{75}$$

depends on the behaviour of μ_{11} all the way to $\tau = 0$, as does

$$\overline{[X_1\,(t) - X_1\,(t')]^2} = \overline{X_1^2}\,(t-t'). \tag{76}^*$$

H. — Lin's Theory

Lin (1960 I, II) has presented a new theory which has some resemblance to a Kolmogorov type of approach, although he feels that it is basically less restrictive. We shall go through his single particle analysis here, filling in some steps.

He chooses as unknown function the mean square Lagrangian velocity difference :

$$\mathbf{v}\,(\mathbf{a}, t+\tau) - \mathbf{v}\,(\mathbf{a}, t) = \int_t^{t+\tau} \alpha\,(\mathbf{a}, t_1)\,dt_1, \tag{77}$$

where α is acceleration, with autocorrelation

$$\alpha\,(\mathbf{a}, t+\tau)\cdot\alpha\,(\mathbf{a}, t) = \alpha^2\,R_0\,(\tau). \tag{78}$$

It is easily shown that

$$\frac{1}{2}\frac{d}{d\tau}\,\overline{|\mathbf{v}\,(t+\tau) - \mathbf{v}\,(t)|^2} = \alpha^2\int_0^{\tau} R_0\,(s)\,ds. \tag{79}$$

For the approximation, he postulates the existence of a time τ_1, such that for

$$\left.\begin{aligned} &\tau_1 < \tau < \infty, \\ &\left|\int_{\tau_1}^{\tau} R_0\,ds\right| \ll \left|\int_0^{\tau_1} R_0\,ds\right|. \end{aligned}\right\} \tag{80}$$

If such a τ_1 exists, (79) can be approximated for $\tau_1 < \tau < \infty$ by

$$\frac{1}{2}\frac{d}{d\tau}\,\overline{|\mathbf{v}\,(t+\tau) - \mathbf{v}\,(t)|^2} \approx B_0, \tag{81}$$

* This form can be deduced by a bit of manipulation. It includes a form for the displacement autocorrelation $X_1\,(t)\,X_1\,(t+\tau)$.

where

$$B_0 = \text{constant} = \alpha^2 \int_0^{\tau_1} R_0 \, ds. \tag{81a}$$

(81) integrates at once to

$$\overline{|\mathbf{v}(t+\tau) - \mathbf{v}(t)|^2} \approx 2 B_0 \tau. \tag{82}$$

This is precisely the same form as (52), but obtained by postulating (80), whose nature must be investigated further. B_0 has the same dimensions as ε, but its « definition », (81*a*) is difficult to interpret.

First of all, it can be shown that

$$\int_0^{\infty} R_0(s) \, ds = 0, \tag{83}$$

so (80) cannot be true for indefinitely large τ.

Another view of Lin's requirement, equation (80), can be gotten by writing the left side of (79), which is exact, in terms of the velocity correlation coefficient,

$$S_0(\tau) \equiv \frac{1}{v^2} \overline{\mathbf{v}(t+\tau) \cdot \mathbf{v}(t)} :$$

$$\frac{\overline{v^2}}{\alpha^2} S'_0(\tau) = \int_0^{\tau} R_0 \, d\sigma, \tag{84}$$

where the prime denotes differentiation. Formally, we also have

$$\frac{\overline{v^2}}{\alpha^2} S'_0(\tau_1) = \int_0^{\tau_1} R_0 \, d\sigma, \tag{85}$$

so (80) can be replaced by

$$|S'_0(\tau) - S'_0(\tau_1)| \ll |S'_0(\tau_1)|. \tag{86}$$

This condition must have a small range of validity for any smooth S_0 function. But it can evidently extend over a broad range of $\tau > \tau_1$ only for a linear $S_0(\tau)$ region, as in (82) and (52). The question to be answered is whether (80) is stronger than, weaker than, or equivalent to the Kolmogorov restriction to very large Reynolds number.

I. — Independance Hypothesis

The Lagrangian velocity autocorrelation is basic to dispersion, as indicated in (26). We are especially interested in

$$\mu_{ik}(o, \tau) \equiv \overline{v_i(\mathbf{a}, t) \, v_k(\mathbf{a}, t+\tau)} = \overline{v_i(o, o) \, v_k(o, \tau)}, \tag{87}$$

i.e.

$$\mu_{ik}(o, \tau) = \overline{u_i[o, o] \, u_k[\mathbf{X}(o, \tau), \tau]}, \tag{88}$$

which can also be written as

$$\mu_{ik}(o, \tau) = \iiint_{-\infty}^{\infty} \overline{u_i(o, o) \, u_k(\mathbf{x}, \tau) \, \delta[\mathbf{x} - \mathbf{X}(o, \tau)]} \, d\mathbf{x}. \tag{89}^*$$

As mentioned earlier, we would like to express μ_{ik} in terms of Eulerian moments. Formally, μ_{11} of (77) depends on the functional probability of $\mathbf{u}(\mathbf{x}, t)$, because the $\mathbf{X}$ is a

* This form due to SAFFMAN (private communication).

functional of $\mathbf{u}$ as indicated by (25). An *ad hoc* simplification can be made by ignoring the analytical connection of (25) and treating $\mathbf{u}, \mathbf{u}'$ $[\equiv \mathbf{u}(\mathbf{x}, \tau)]$ and $\mathbf{X}$ as randomly related only. μ_{ik} *is* a moment of their joint probability density function in all cases :

$$\mu_{ik}(o, \tau) = \iiiint_{-\infty}^{\infty} \left\{ \iiint_{-\infty}^{\infty} u_i u'_k \, \delta(\mathbf{x} - \mathbf{X}) \, d\mathbf{x} \right\} \mathrm{P}(u_i, u'_k, \mathbf{X}) \, du_i \, du'_k \, d\mathbf{X}. \quad (90)$$

In the limit of very large τ, there is no reason to expect a statistical connection between $\mathbf{X}$ and $\mathbf{u}$, so

$$\mathrm{P}(u_i, u'_k, \mathbf{X}) \to \mathrm{P}_{u_i u'_k} \mathrm{P}_{\mathbf{X}}(\mathbf{X}). \quad (91)$$

This transforms (90) to

$$\mu_{ik}(o, \tau) \to \iiint_{-\infty}^{\infty} \mathrm{E}_{ik}(\mathbf{X}, \tau) \, \mathrm{P}_{\mathbf{X}}(\mathbf{X}) \, d\mathbf{X} \quad (92)^*$$

$\mathrm{E}_{ik}(\mathbf{r}, \tau)$ is the Eulerian correlation tensor.

Empirically it is found that X_1, X_2, X_3 are normal and uncorrelated in grid-generated (nearly isotropic) turbulence for all times. It would be interesting to combine this fact with the FAVRE-GAVIGLIO-DUMAS (1953) data on Eulerian space-time correlation, to compare (92) with Lagrangian data for the full time range.

The approach to a form like (92) can be displayed more clearly by a more rudimentary case. Suppose we have two stationary random variables, $f(x)$ and $s(x)$, which are in general correlated. One of the attributes of a turbulent Lagrangian correlation also appears in the f autocorrelation with random separation s :

$$\mathrm{L} \equiv \overline{f(x) \, f(x + s)}. \quad (93)$$

The series expansion is

$$\mathrm{L} = \overline{f^2} + \overline{ff' \, s} + \frac{1}{2!} \overline{ff'' \, s^2} + \frac{1}{3!} \overline{ff''' \, s^3} + \ldots\ldots \quad (94)$$

Suppose now that we make s uncorrelated with f and its derivatives. Then, taking $\bar{s} \equiv 0$ for simplicity,

$$\mathrm{L} = \overline{f^2} + \frac{1}{2!} \overline{ff'' \, s^2} + \frac{1}{3!} \overline{ff''' \, s^3} + \ldots\ldots, \quad (95)$$

which is easily shown to be

$$\mathrm{L} = \int_{-\infty}^{\infty} \mathrm{E}(s) \, \mathrm{P}_s(s) \, ds. \quad (96)$$

P_s is the probability density function of s; $\mathrm{E}(r) \equiv \overline{f(x) \, f(x + r)}$, the « Eulerian type » of autocorrelation.

The turbulence case is, of course, one in which s is an integral function of f. Whenever a more explicit connection exists, more detailed results can be deduced. Work on these « self-dependent functions » is in progress.

J. — Deissler's Computation for Very Small Reynolds Number

In the latest of a series of isotropic spectral computations made determinate by neglecting the $(n + 1)$ -order moments in the equation for the n^{th} -order ones, DEISSLER (1961) has appended a computation of turbulent dispersion. This is based on the further

* CORRSIN (1959).

assumption [suggested by BURGERS (1951)] that at very small Reynolds numbers the Lagrangian and Eulerian velocity correlations in time only may be approximately equal.

The validity of Deissler's procedure for rendering the hierarchy of Eulerian moment equations determinate will doubtless be discussed at another session of this symposium. It should be noted only that his choice of a function of integration fails to introduce the singular term $k^5 \log k$, required by the analysis of BATCHELOR and PROUDMAN (1956)*. Also, he requires the fourth moment of the velocity correlation to be independent of time, a prediction of LOITSIANSKII (1939) about which BATCHELOR and PROUDMAN raise doubts.

Nevertheless, these two simplifications are presumably valid for very small turbulence Reynolds number, the « final period » of decay, for which the double correlations alone are retained. This may be why no dramatic paradoxes appear in the double space-time correlations in the « almost final period », obtained by neglecting quadruple correlations in the triple correlation equation.

It is more appropriate in this paper to look at the BURGERS approximation,

$$\mu_{11}(o, \tau) \equiv \overline{v_1(\mathbf{a}, t)\, v_1(\mathbf{a}, t+\tau)} \approx \overline{u_1(\mathbf{x}, t)\, u_1(\mathbf{x}, t+\tau)}. \tag{97}$$

This corresponds to neglecting all convective terms in (34), (35) and corresponding higher order derivative moments :

$$\overline{\left(\frac{\partial v_1}{\partial t}\right)^2} \approx \overline{\left(\frac{\partial u_1}{\partial t}\right)^2}, \tag{98}$$

$$\overline{\left(\frac{\partial^2 v_1}{\partial t^2}\right)^2} \approx \overline{\left(\frac{\partial^2 u_1}{\partial t^2}\right)^2}, \tag{99}$$

etc. At first glance (98) as approximation to (34) looks like a traditional « Stokes flow » type of approximation, known to be successful for estimating the flow fields past three-dimensional solid bodies at Reynolds numbers well below unity. If we use (13) to write the exact relationship, however,

$$\overline{v_1(o, o)\, v_1(o, \tau)} = \overline{u_1[o, o]\, u_1[\mathbf{X}(o, \tau), \tau]} \tag{100}$$

we see that (97) neglects particle displacement — which suggests further scrutiny.

Before proceeding, it must be pointed out that this analysis gives fair agreement with the dispersion data of UBEROI and CORRSIN (1953), especially for the lower REYNOLDS number region, $R_\lambda \approx 20$ [see figure 4].

From his final period solution, with numerical coefficient taken from decay experiments, DEISSLER finds

$$\sqrt{\overline{u_1^2}}\, \alpha_E = 0\,(\lambda). \tag{101}$$

α_E is the Eulerian « time microscale » :

$$\frac{1}{\alpha_E} \equiv \frac{1}{\overline{u_1^2}} \overline{\left(\frac{\partial u_1}{\partial t}\right)^2}. \tag{102}$$

On the other hand, the large R_λ estimates for $\dfrac{\lambda}{\sqrt{\overline{v_1^2}}\, \alpha_{11}}$ extrapolate to ∞ for $R_\lambda \to o$**. Equation (98) says that $\alpha_E \approx \alpha_{11}$. The question seems unsettled. PERHAPS (98) is not

* k is wave number.

** The small R_λ estimate of this quantity by UBEROI and CORRSIN (1953) is probably wrong because it neglects the pressure term relative to the viscous term [using (36) as starting point].

a straightforward Stokes approximation. In the ordinary justifications, the convective terms are neglected relative to viscous terms, not time-derivative terms.

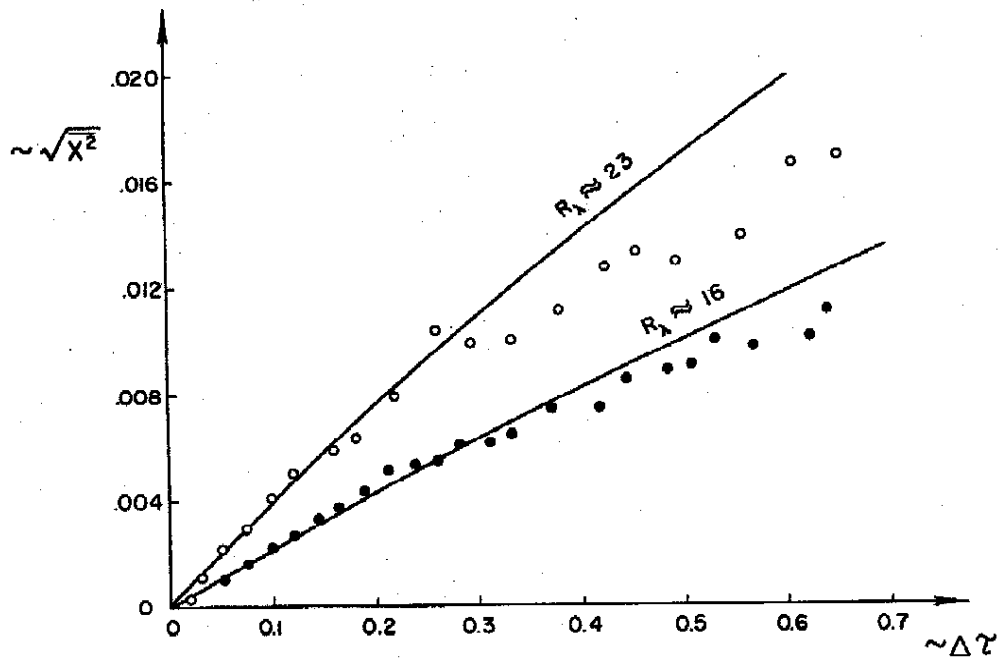

FIGURE 4
DEISSLER (1961) correlation discard theory; comparisons with experiment [UBEROI & CORRSIN (1953)]

In closing this section, we should recall a qualitative conjecture on the difference between Eulerian and Lagrangian correlations in time* : it seems likely that

$$\overline{u_1(o,o)\,u_1(o,t)} \leqslant \overline{v_1(o,o)\,v_1(o,t)}, \tag{103}$$

where $u_1(\mathbf{x},t)$ and $v_1(\mathbf{a},t)$ are corresponding Eulerian and Lagrangian component fields. $u_1(o,o) = v_1(o,o)$, since we start at the origin at time o. But $v_1(o,t)$ follows along the same fluid material, whereas $u_1(o,t)$ is the velocity of other fluid, swept past the origin by the general turbulent agitation. (103) is based on the notion that the velocity persistence of a particular fluid particle should last longer than any explicit Eulerian choice.

One extreme case is the random (rigid body) translation of a fluid. Then the two correlations are equal.

K. — Recent Analyses

Still unpublished, but available in report form are application by ROBERTS (1960) and KRAICHNAN (1961) of the KRAICHNAN and cumulant-discard turbulence theories, respectively, to dispersion. The former leads to a wave-like dispersion in first approximation, the latter to negative probabilities of displacement. The latter result is primarily a severe blow to the cumulant-discard hypothesis. In the same report, KRAICHNAN

* Made by SHEPPARD and by BATCHELOR at the Oxford Symposium in 1958.

demonstrates a small Péclet number series expansion approach, somewhat akin to DEISSLER's approximation.

Both OBUKHOV (1959) and LIN [(in press) and M. I. T. thesis by CHADAM (1961)] have proposed the exploitation of the similarities between Brownian motion and turbulent diffusion. Beginning with TAYLOR (1921), most basic work has concentrated on the differences.

II. — RELATIVE DISPERSION OF TWO PARTICLES

A. — The Problem and the Eulerian Formulation

The relative dispersion of two fluid material points is a direct generalization of single particle dispersion. For the mean « concentration » field (e. g., in Eulerian frame) we need only superpose the two concentration fields of the two « sources ». These are just the two probability density functions of single particle displacement. But now we also seek the *joint probability density function* of the two particle displacements (Lagrangian frame) or the Eulerian equivalent, the correlation or coincidence rate for the concentration at two points in space-time.

The simplest goal of the case without molecular diffusion is perhaps the time history of the mean square separation distance.

The diffusion equation, (1) or (42), still applies; the initial condition is

$$\Theta(\mathbf{x}, o) = \delta(\mathbf{x}) + \delta(\mathbf{x} - \mathbf{A}), \tag{104}$$

when one particle is released at the origin, $\mathbf{x} = o$, and the other at any other position, $\mathbf{x} = \mathbf{A}$. Of course, (104) would also be the initial condition for the non-diffusive case, equation (8).

Up to the present, there seems to have been no effort put into the Eulerian approach.

B. — Lagrangian Kinematic Formulation

BRIER (1950) and BATCHELOR (1952) were apparently the first to publish direct applications of TAYLOR's (1921) formulation to the relative dispersion problem.

With restriction to stationary, homogeneous turbulence and zero mean velocity, each of the two fluid material points disperses according to equation (27) relative to its own starting point. We write

$$\mathbf{Y}(\mathbf{A}, t) \equiv \mathbf{X}(\mathbf{a} + \mathbf{A}, t) - \mathbf{X}(\mathbf{a}, t), \tag{105a}$$

or

$$\mathbf{Y} \equiv \mathbf{X}' - \mathbf{X}. \tag{105b}$$

Picking $\mathbf{a} = o$ for simplicity, we have as x_1 components

$$\left.\begin{aligned} X_1' &= A_1 + \int_0^t v_1(\mathbf{A}, t')\, dt' \\ X_1 &= \int_0^t v_1(o, t')\, dt' \end{aligned}\right\} \tag{106}$$

Thus,

$$Y_1 = A_1 + \int_0^t (v_1' - v_1)\, dt', \tag{107}$$

so the expression corresponding to (26) is

$$\frac{1}{2}\frac{d}{dt}\overline{Y_1^2} = \int_0^t \rho_{11}(\tau, t; \mathbf{A})\, d\tau \tag{108}$$

where ρ_{ik} is the Lagrangian autocorrelation of the *difference velocity*, $\mathbf{w} \equiv \mathbf{v}' - \mathbf{v}$:

$$\rho_{ik}(\tau, t; \mathbf{A}) = \overline{w_i(\mathbf{A}, t)\, w_k(\mathbf{A}, t+\tau)}. \tag{109}$$

Since $\mathbf{w}(\mathbf{A}, t)$ cannot be a stationary variable except for asymptotically large t*, ρ_{ik} must depend on t as well as τ. Therefore a transformation analogous to that which produces (27) is not possible.

Equation (108) is effectively a « diffusivity », insofar as that concept may seem definable in this problem. It has, in any case, a variety of special asymptotes corresponding to cases in which $\mathbf{A}$, t or τ each tends toward a small or large limit, singly or in pairs.

We shall not review this array of asymptotic forms in this paper. It will suffice to make the following remarks :

(1) When $t \to o$, we get a simple linear growth of $\frac{d}{dt}\overline{Y_1^2}$, with a coefficient depending on Eulerian velocity correlation at fixed time.

(2) When $t \to \infty$ for any $\mathbf{A}$, or when $\mathbf{A} \to \infty$ for any t, the two material points wander independently, so (108) degenerates to twice (26).

C. — Similarity Theory; The Obukhov and Batchelor Derivations of « Richardson's Law ».

In one respect relative dispersion is more accessible to theoretical analysis than is single particle dispersion : when two particles in large Reynolds number turbulence are separated by a distance falling within the locally isotropic range of eddy sizes, the « diffusivity » can be deduced by postulating similarity in the Kolmogorov sense and using dimensional reasoning.

The particular goal of most theories on the subject of relative dispersion has been to derive Richardson's (1926) empirical law for « diffusivity »,

$$\frac{d\overline{Y_1^2}}{dt} \sim \overline{Y_1^2}^{2/3}. \tag{110}$$

Suppose that the initial separation $|\mathbf{A}|$ is of the order of the Kolmogorov microscale, $\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$, or smaller (but not zero). $|\mathbf{A}|$ can be chosen so that after a time very long compared with the characteristic time scales of the turbulence, but not infinite, the two particles will satisfy the double inequality

$$\eta \ll \sqrt{\overline{Y^2}} \ll L, \tag{111}$$

where L characterizes the turbulent large structure.

* The limit depends on $\mathbf{A}$, being larger for smaller $\mathbf{A}$.

It must first be shown that whenever two particles are separated by any $\sqrt{\overline{Y^2}}$, the further statistical increase in distance is governed by turbulent eddies of size comparable with $\sqrt{\overline{Y^2}}$. This is ordinarily taken to be obvious. It can't be proven, but can perhaps be made plausible by consideration of the mean square relative velocity,

$$\overline{w_1^2} = \overline{(v_1' - v_1)^2}$$
$$\therefore \overline{w_1^2} = 2\overline{v_1^2} - 2\overline{v_1 v_1'} \qquad (112)$$
$$\text{since} \quad \overline{v_1'^2} = \overline{v_1^2}.$$

Of course $\overline{w_1^2}$ does *not* actually indicate tendencies to separate [at the moment of tagging, for example, $\overline{w_1^2} \neq o$ even though $\frac{d}{dt}\overline{Y_1^2} = o$; Batchelor (1952)]. It does, however, indicate the intensity of relative agitation, usually monotonic with dispersion in a random system.

From reasoning like that in the « Independance Hypothesis » (Section I.I), we can guess that $\overline{w_1^2}$ mostly increases with $\sqrt{\overline{Y^2}}$ as sketched in figure 5. If $\sqrt{\overline{Y^2}}$ is smaller than most of the turbulence structure, the two fluid material points have almost equal velocities. If $\sqrt{\overline{Y^2}}$ is larger than most of the turbulence structure, their velocities are uncorrelated. For any particle separation, this implies that the eddies most efficient in dispersing are those smaller than the separation length. Efficiency here means diffusivity contribution per unit of kinetic energy.

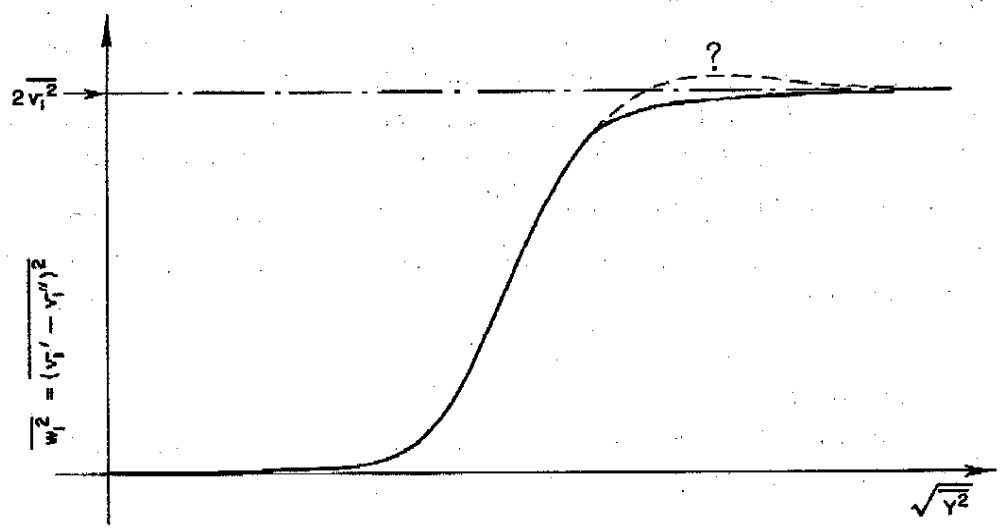

Figure 5

Although the smallest eddies are most « efficient », they contain relatively little energy. The large eddies (larger than the particle separation) contain most of the energy, but they are very « inefficient ». Hence, we arrive at the conclusion that for ordinary turbulence, the principal contribution to relative dispersion comes from eddies of a scale the same order of magnitude as the particle pair separation, figure 6*.

* It would clearly be desirable to make the foregoing discussion « honest » by repeating for particle pairs the Fourier analytical representation applied to single particles in equation (40).

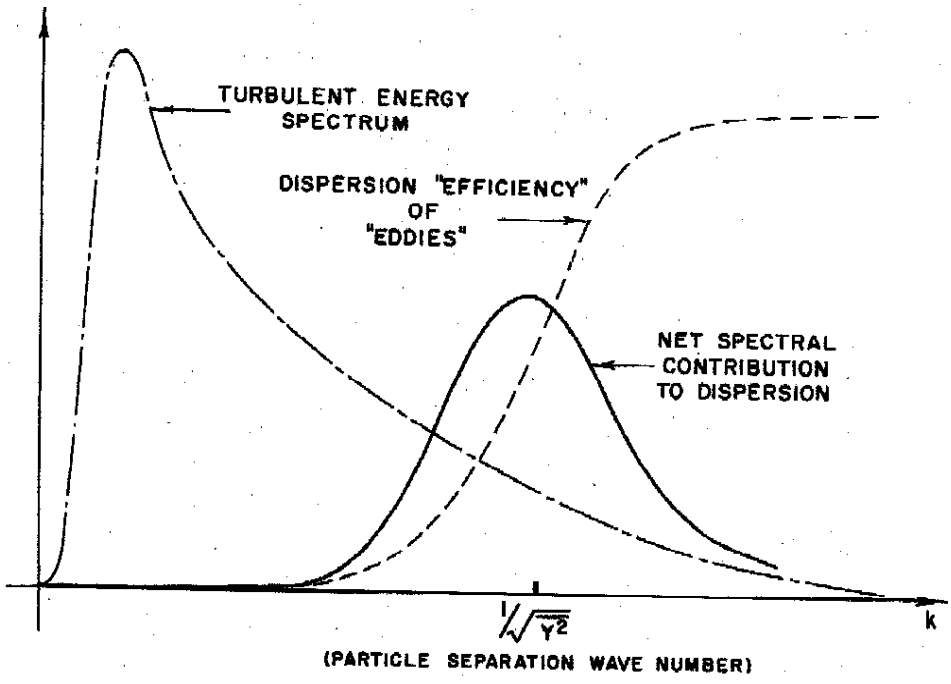

FIGURE 6

Having arrived at this conclusion, we are now free to argue that if the particle separation lies in the KOLMOGOROV similarity range, the statistical properties of their relative motion must be subject to the corresponding similarity reasoning.

The earliest and most direct route to RICHARDSON's Law for the dependence of « diffusivity » on particle separation was that of OBUKHOV (1941). Assuming that $\overline{Y^2}$ is within the *inertial*, locally isotropic range, he postulated that $\frac{1}{2}\frac{d}{dt}\overline{Y^2}$ depends only on $\overline{Y^2}$ and on the total dissipation rate ε. Then the only dimensionally possible relation for « diffusivity » is

$$\frac{1}{2}\frac{d\overline{Y^2}}{dt} \sim \varepsilon^{1/3}\overline{Y^2}^{2/3}, \tag{113}$$

which is RICHARDSON's LAW.

BATCHELOR (1949, 1952) began a bit more generally. Assuming that $\sqrt{\overline{Y^2}}$ is within the locally isotropic range, he postulated

$$\frac{1}{2}\frac{d\overline{Y^2}}{dt} = f_n(\varepsilon, \nu, A; t). \tag{114}$$

The choice of A and t as parameters is more or less equivalent to OBUKHOV's choice of $\sqrt{\overline{Y^2}}$, but it is pedagogically more appealing because A and t are more easily fixed by the experimenter.

The dimensional consequence of (114) is

$$\frac{1}{2}\frac{d\overline{Y^2}}{dt} = \nu\, G\left[\left(\frac{\varepsilon}{\nu^3}\right)^{1/4} A, \left(\frac{\varepsilon}{\nu}\right)^{1/2} t\right], \tag{115}$$

with G an arbitrary function.

When $\sqrt{\overline{Y^2}}$ is further restricted to be in the inertial isotropic range, the explicit dependence on ν should vanish. Taking G to be a product of powers of its two dimensionless arguments, BATCHELOR eliminated one of these exponents, leaving an arbitrary function of one argument :

$$\frac{1}{2}\frac{d\overline{Y^2}}{dt} = \varepsilon G_1\left(\frac{A}{\varepsilon^{1/2} t^{3/2}}\right). \tag{116}$$

BATCHELOR next restricted the problem to « large t » in the sense that

$$\sqrt{\overline{Y^2}} \gg A, \tag{117}$$

maintaining the preceding restriction that $\sqrt{\overline{Y^2}}$ lie in the inertial, isotropic range. This implies at least

$$A \ll \sqrt{\overline{Y^2}} \ll L, \tag{118}$$

where L is an appropriate choice of integral length scale. In order to fulfill (118) at REYNOLDS numbers attainable in the laboratory, it may be necessary to make the initial particle spacing A as small as the KOLMOGOROV microscale $\left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$.

In any case, with (117) it is reasonable to require that (116) be independent of A. Taking G_1 to be a power law, the only choice is the zero power, so that

$$\frac{1}{2}\frac{d\overline{Y^2}}{dt} \sim \varepsilon t^2. \tag{119}$$

Solving this for

$$\overline{Y^2} \sim \varepsilon t^3, \tag{120}$$

and using this to replace t in (119), BATCHELOR arrived at RICHARDSON'S Law, equation (113).

Another interesting result follows in the $t \to 0$ limit *if* $\sqrt{\overline{Y^2}}$ (hence A) is in the inertial isotropic range. From the fact that accelerations cannot be infinite, we know that the velocity correlations have zero slope for $t = 0$. Hence [analogous to the first term in equation (32) for a single particle], we know from (108) that for $t \to 0$,

$$\frac{d\overline{Y^2}}{dt} \to \sim t \tag{121}$$

Comparing this with (116), we obtain

$$\frac{1}{2}\frac{d\overline{Y^2}}{dt} \to \sim \varepsilon^{2/3} A^{2/3} t. \tag{122}$$

Up to the present time, there has been very little experimentation on relative dispersion in controlled « simple » laboratory turbulence. The measurements of KISTLER (1956) cover too small a time range and REYNOLDS number for comparison with similarity theory.

D. — Lin's Theory

Lin applied his method (see section I, H.) to the relative dispersion problem also. His principal assumptions are

(a) t is large enough that the initial separation, **A**, and initial relative velocity, **w** (*o*), can both be ignored. This is the same as one of BATCHELOR'S conditions.

(b) t is much larger than some finite reference time τ_1, defined by the inequalities

$$\left| \int_{\tau_1}^{t} \mathrm{R}(\sigma)\, d\sigma \right| \ll \left| \int_{0}^{\tau_1} \mathrm{R}(\sigma)\, d\sigma \right|, \tag{123a}$$

and

$$\left| \int_{\tau_1}^{t} \sigma\, \mathrm{R}(\sigma)\, d\sigma \right| \ll \left| \int_{0}^{\tau_1} \sigma\, \mathrm{R}(\sigma)\, d\sigma \right|, \tag{123b}$$

where R is the autocorrelation function of the *relative acceleration* of the two particles :

$$\mathrm{R}(\tau, t; \mathbf{A}) \equiv \overline{\mathbf{w}_t(\mathbf{A}, t) \cdot \mathbf{w}_t(\mathbf{A}, t-\tau)}. \tag{124}$$

The subscript denotes partial derivative. The argument in (123) is τ.

Following a TAYLOR procedure, but neglecting $\mathbf{w}(\mathbf{A}, o)$, he deduces

$$\frac{d}{dt}\overline{w^2} = 2\int_0^t \mathrm{R}(\tau, t; \mathbf{A})\, d\tau. \tag{125}$$

Lin then assumes that R may be approximated as a function of τ and $\mathbf{A}$ only, hence that $\mathbf{w}_t(\mathbf{A}, t)$ is a stationary random avriable. With this assumption he is able to deduce RICHARDSON'S Law as

$$\frac{1}{2}\frac{d\overline{\mathrm{Y}^2}}{dt} \approx \left(\frac{3}{2}\right)^{2/3} \mathrm{B}\left(\overline{\mathrm{Y}^2}\right)^{2/3}, \tag{126}$$

instead of OBUKHOV'S form, equation (113). B is a « constant » having the dimensions of energy dissipation rate :

$$\mathrm{B} \equiv \overline{w_t^2} \int_0^{\tau_1} \mathrm{R}\, d\tau. \tag{127}$$

(126) has the apparent advantage of a fully estimated coefficient, if $\mathrm{R}(\tau; \mathbf{A})$ is known. Lin also claims that his derivation requires no restriction to very large REYNOLDS numbers. As we have seen from the one-particle case, however, large REYNOLDS number could conceivably be implied by (123*a*, *b*).

Furthermore, the assumption of statistically stationary relative acceleration seems drastic indeed.

E. — Independence Hypothesis

For separation distances sufficiently large that the two particles move almost independently, we may expect that the two-particle Lagrangian velocity correlation μ_{ik} can be approximated as a separation-probability-weighted average over the Eulerian velocity correlation field, $\mathrm{E}_{ik}(r, \tau)$.

For the idealized case of homogeneous, stationary turbulence, this generalization of equation (92) can be written by inspection as

$$\mu_{ik}(\mathbf{A}, \tau, t) = \iiint_{-\infty}^{\infty} \mathrm{E}_{ik}(\Delta, \tau)\, \mathrm{P}_\Delta(\Delta; \mathbf{A}, \tau, t)\, d\Delta, \tag{128}$$

where Δ is the separation of the two particles at different times,

$$\Delta \equiv \mathbf{X}(\mathbf{a} + \mathbf{A}, t + \tau) - \mathbf{X}(\mathbf{a}, t). \tag{129}$$

For the special case of particle velocities at the same time,

$$\Delta \to \mathbf{Y} \equiv \mathbf{X}(\mathbf{a}+\mathbf{A}, t) - \mathbf{X}(\mathbf{a}, t):$$

$$\mu_{ik}(\mathbf{A}, o, t) = \iiint_{-\infty}^{\infty} \mathrm{E}_{ik}(\mathbf{Y}, o)\, \mathrm{P}_{\mathrm{Y}}(\mathbf{Y}; \mathbf{A}, t)\, d\mathbf{Y}. \tag{130}$$

As for the single particle case, it may be worthwhile to test (130) as an *ad hoc* estimate for separations $\sqrt{\overline{\mathrm{Y}^2}}$ not asymptotically large. The probability density functions for isotropic turbulence can doubtless be assumed to be normal.

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SOME ASPECTS OF THE EFFECTS OF THE MOLECULAR DIFFUSIVITY IN TURBULENT DIFFUSION

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SOMMAIRE

L'effet de la diffusion moléculaire sur la propagation et la dispersion d'une quantité scalaire passive transportée par un champ de vitesse turbulent sera discuté.

La diffusion de taches de chaleur en écoulement turbulent isotrope est considéré d'abord. Pour de faibles valeurs du temps s'écoulant depuis leur formation, certaines propriétés des taches de chaleur peuvent être facilement évaluées. En particulier, on peut montrer que l'interaction du processus de diffusion moléculaire avec l'effet de dispersion de la turbulence, *augmente* le taux de refroidissement de la tache, ou aussi bien sa dimension, mais que cette interaction *diminue* la dispersion de la chaleur relative au point d'émission.

En second lieu l'effet de la diffusion moléculaire sur l'étendue du sillage derrière une source linéaire en turbulence homogène est étudié. (Ceci est le problème de la diffusion d'une particule). La description de la dispersion dans les termes d'une fonction d'auto-corrélation de Lagrange généralisée est discutée, et la dispersion pendant de courtes périodes de temps est calculée. On trouve que le carré moyen de la largeur du sillage est *inférieur* à ce qu'il serait si les processus de diffusion moléculaire et turbulente étaient indépendants et additifs.

On considère la signification du travail de Townsend, qui montre que, pour de courts intervalles de temps, la largeur instantanée du sillage est en moyenne augmentée (ou de manière équivalente, que le taux de refroidissement d'un sillage chauffé est initialement accéléré) par cette interaction.

On discute de l'estimation de la dispersion, pour de longues périodes de temps, et on la compare avec les données expérimentales limitées dont on dispose.

Le problème du calcul de cette interaction lorsque le nombre de Prandtl est très petit, et l'effet de la diffusion moléculaire sur les taux moyens de transfert, dans une situation quasiment stationnaire, seront mentionnés.

SUMMARY

The effect of molecular diffusion on the spreading and dispersion of a passive scalar quantity convected by a turbulent velocity field will be discussed. First, the diffusion of heat spots in isotropic turbulence is considered. For small values of the time from formation, certain properties of the spot can be evaluated easily. In particular, it can be shown that the interaction of the process of molecular diffusion with the stretching effect of the turbulence *increases* the rate of cooling of the spot or equivalently its size, but that the interaction *decreases* the spreading of the heat relative to the point of release.

Second, the effect of molecular diffusion on the spreading of the wake behind a line source in homogeneous turbulence is considered. (This is the one-particle diffusion problem.)

The description of the dispersion in terms of a generalised Lagrangian auto-correlation function is discussed and the dispersion for small times is calculated. It is found that the mean-square width of the wake is *less* than it would be if the processes of molecular and turbulent diffusion were independent and additive. The significance of TOWNSEND's work which shows that, for small times, the instantaneous wake width is on average increased (or equivalently the rate of cooling of a heated wake is initially accelerated) by the interaction is considered. An estimate of the spreading for large times is discussed and compared with the limited available experimental data.

The problems of calculating the interaction when the Prandtl number is very small and of the effect of molecular diffusion upon mean rates of transport in a quasi-steady situation will be mentioned.

1. — Introduction

When a cloud of material is released in a turbulent flow, it is spread out and dispersed by the action of the process of turbulent diffusion. It is often useful to think of this dispersion in terms of two of its properties. These are (i) the random wandering of the cloud as a whole and (ii) the change in shape of the cloud. The analysis of property (i) is the problem of determining the probability of locating marked fluid at a particular point. This question has been analysed by BATCHELOR (see [1]), who showed it was equivalent to determining the statistical behaviour of the displacement of a single fluid particle. An experimental configuration convenient for the investigation of property (i) is the spreading of the wake behind a source in a turbulent stream. The investigation of property (ii) is more difficult, and the analysis has so far been mainly restricted to the problem of the separation of two particles (see [1]).

In reality, any convected quantity will also be dispersed by molecular diffusion which transports material down a concentration gradient. Both properties (i) and (ii) are affected in some way by this additional transport of material, and the total dispersion is due to the combined effect of molecular diffusion and the diffusion due to the turbulent velocities.

In a steady (or quasi-steady) configuration, where the mean concentration is a function of position but is independent of (or varies very slowly with) the time, the mean transport of material down the gradient of mean concentration is often described in terms of an eddy diffusivity. This eddy diffusivity depends upon the (not necessarily local) properties of the turbulent field, but is also a function of the molecular diffusivity.

In practice, the effect of the molecular diffusion in turbulent diffusion is usually small and is in effect masked by the factor of ignorance about the turbulence. Nevertheless, it is useful to know something about it, if only to be able to justify its neglect.

Molecular diffusion does not act only to modify the rate of diffusion of material. It is known that the turbulent motion distorts a cloud into a more and more highly irregular and contorted shape. The curvature of the surfaces of constant concentration increases and their distance apart decreases, and the cloud becomes more and more " spotty ", until the concentration gradients are large enaugh for molecular diffusion to become significant. The molecular diffusion then acts to smooth out the " spottiness ", and make the spatial distribution more uniform until a balance is reached, see [1] [2] and [3].

2. — The diffusion of heat spots

Considerable insight into the effect of molecular diffusion can be obtained by considering the diffusion of a small heat spot created by the liberation of a finite amount of heat at a point. Let $\theta(\mathbf{x}, t)$ denote the temperature (or concentration of any other material quantity), then

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \nabla \theta = \kappa \nabla^2 \theta, \tag{1}$$

where κ is the molecular diffusivity and $\mathbf{u}(\mathbf{x}, t)$ the turbulent velocity field. The fact that equation (1) is linear in θ is of considerable importance (it is assumed that all concentrations are sufficiently small to have no effect on the turbulence), since it allows the linear superposition of solutions.

Townsend [4] has shown how the shape of the spot for small values of the time from release can be calculated. Referred to axes moving with the spot and in the direction of the principal rates of strain, equation (1) is approximately

$$\frac{\partial \theta}{\partial t} + \alpha x \frac{\partial \theta}{\partial x} + \beta y \frac{\partial \theta}{\partial y} + \gamma z \frac{\partial \theta}{\partial z} = \kappa \nabla^2 \theta, \tag{2}$$

where α, β, γ are the principal rates of strain whose variation with time may be neglected for small changes of the time. If the heat is released at time $t = 0$, the solution of (2) is

$$\theta = \theta_m(t) \exp\left[-\frac{1}{2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\right], \tag{3}$$

where

$$a^2 = \frac{\kappa}{\alpha}(e^{2\alpha t} - 1), \quad b^2 = \frac{\kappa}{\beta}(e^{2\beta t} - 1), \quad c^2 = \frac{\kappa}{\gamma}(e^{2\gamma t} - 1), \tag{4}$$

and

$$\theta_m(t) = \frac{A}{abc} = \frac{A}{(2\kappa t)^{3/2}}\left(1 - \frac{\alpha^2 + \beta^2 + \gamma^2}{12} t^2 + \ldots\right). \tag{5}$$

The heated volume is stretched by the turbulent shear into either a long filament or a flat disk, according as $\alpha\beta\gamma$ is positive or negative. This distortion increases the concentration gradients and accelerates the spreading of the heat so that the heat is more widely distributed throughout the fluid. This is represented by the increased rate at which the maximum temperature drops compared with the rate for a spot in a fluid at rest. The mean dispersion of the spot (or variance of the temperature) in an arbitrary direction is, relative to its centroid,

$$\begin{aligned}\frac{1}{3}\langle a^2 + b^2 + c^2 \rangle &= 2\kappa t + \frac{4}{9}\kappa t^3 \langle \alpha^2 + \beta^2 + \gamma_2 \rangle + \ldots \\ &= 2\kappa t + \frac{2}{9}\kappa t^3 w^2 + \ldots,\end{aligned} \tag{6}$$

where the bracket denotes an average over the ensemble of turbulent realizations, w^2 is the mean-square-vorticity, and the shearing components of the turbulence are supposed statistically isotropic. Thus the size of the spot is increased by the turbulence.

TOWNSEND (see [1]) has also considered the increase in width of the wake behind a line source due to the stretching and rotation of the sheat of heated fluid by the turbulence. Neglecting the wandering of the wake as a whole and considering only its instantaneous thickness, it can be shown that the mean-square instantaneous wake width increases like

$$2\kappa t + \frac{5}{9}\kappa t^3 w^2 + \ldots \qquad (7)^*$$

which implies further that the rate of cooling of the wake is accelerated by the turbulence.

The implication of these results is that the rate at which the volume of heated or marked fluid increases is accelerated by the turbulence. Note that in the absence of molecular diffusion, the volume of marked fluid remains absolutely constant. Turbulent diffusion increases the size of the region in which marked fluid is to be found, but it does this by making the concentration "spotty" or pulling the marked fluid into long thin lines and sheets; it does not increase the number of fluid elements which are marked, this being accomplished by molecular diffusion alone. The rate at which this last process occurs depends upon the concentration gradients. These are on average increased by the stretching due to turbulence, and in this sense we may say that the turbulence accelerates the molecular diffusion. In other words, if a given amount of heat is released, the volume of heated fluid at a later instant is greater if the fluid is in turbulent motion than it is if the fluid is at rest.

The above calculations prove this for small times only, but it is clear from the physical argument that it is a continuing process, ceasing only when the concentration is uniform. This, however, is for from being the whole story, as we shall now see.

Let us consider the dispersion of the heat spot (i. e. the variance of the temperature distribution) relative to a fixed origin; this is essentially the one particle diffusion problem. For small values of the time from release, we can expand the velocity $\mathbf{u}(\mathbf{x}, t)$ in equation (1) as a Taylor series in space and time; thus (in terms of the suffix notation)

$$u_i(\mathbf{x}, t) = (u_i)_0 + \left(\frac{\partial u_i}{\partial x_j}\right)_0 x_j + \left(\frac{\partial u_i}{\partial t}\right)_0 t + \frac{1}{2}\left(\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right)_0 x_j x_k + \ldots$$

Normalise θ so that $\int \theta\, dv = 1$, and define the moments of θ by $X_i = \int x_i \theta\, dv$, $X_{ij} = \int x_i x_j \theta\, dv$, etc. Then differentiating (1) with respect to time, multiplying by a product of space coordinates and integrating by parts over all space, we obtain equations for the time derivatives of the moments in terms of the moments of the same and lower orders. The evaluation of these time derivatives at $t = 0$ gives the coefficients of a Taylor series in time of the moments. (This method was suggested in a letter by

* TOWNSEND obtains the coefficient $\frac{28}{45}$ rather than $\frac{5}{9}$. This is due to the use of an incorrect diffusion equation which does not conserve the total amount of heat. In the notation of [1] equation 3.8 of [1] should be $\frac{\partial \theta}{\partial t} + \frac{\alpha\, \partial(\theta n)}{\partial n} = \frac{\kappa\, \partial^2 \theta}{\partial n^2}$. The result (7) then follows from this equation on applying an identical analysis.

Dr. T. H. Ellison. A somewhat different method was employed in [5].) In this manner, it is found that

$$X_i(t) = (u_i)_0\, t + \frac{1}{2}\left(\frac{\partial u_i}{\partial t}\right)_0 t^2 + \frac{1}{2}\left(u_j \frac{\partial u_i}{\partial x_j}\right)_0 t^2 + \frac{1}{2}\kappa(\nabla^2 u_i)_0\, t^2 + O(t^3). \qquad (8)$$

The first three terms on the right-hand-side give the displacement of the fluid particle which was originally coincident with the heat spot. Denote this displacement by Y_i. The fourth term represents the separation of the centroid of the heat spot and the fluid particle. Now

$$\begin{aligned} \langle X_1^2 \rangle &= \langle Y_1^2 \rangle + \kappa t^3 \langle (u_1 \nabla^2 u_1)_0 \rangle + \kappa O(t^4) \\ &= \langle Y_1^2 \rangle - \kappa t^3 w^2 + \kappa O(t^4) \end{aligned} \qquad (9)$$

if the small scale components of the turbulence are isotropic. Thus the effect of molecular diffusion is to make the dispersion of the centroid, relative to the point of release, less than that of the originally coincident fluid particle. The dispersion of the spot relative to its centroid is given to sufficient accuracy for the present purpose by (6), so that the dispersion (in the 1-direction, say) of the spot relative to the point of release is

$$\langle Y_1^2 \rangle + 2\kappa t - \frac{1}{9}\kappa t^3 w^2 \left\{1 + O\left[\left(1 + \frac{\kappa}{\nu}\right)(w^2)^{1/2}\, t\right]\right\}, \qquad (10)$$

where the error follows from a more careful analysis of the neglected terms. (Alternatively, the result (10) may be obtained by calculating X_{11} in the manner described above.) The physical significance of the error term is that the size of the spot should be small compared with the length scale of the small eddies and the time should be small compared with the time scale of the small eddies.

If the effects of the molecular and turbulent diffusion were independent and additive, the dispersion would be $\langle Y_1^2 \rangle + 2\kappa t$. The extra terms in (10) are due to the interaction, and they represent the effect of the additional concentration gradients produced by the turbulent shear. The striking feature of (10) is that the interaction is negative (in the initial stages at least) and reduces the spreading of the heat relative to a fixed origin. If we ignore the $2\kappa t$ term which is the only term present when the fluid is at rest, we can say that the dispersion decreases as the molecular diffusivity increases, and that the molecular diffusion decelerates the turbulent diffusion.

The physics of the decelerated diffusion can be described as follows. The diffusion of the heat relative to the origin is determined mainly by the convection of the heat by the fluid. Because molecular diffusion increases the volume of heated fluid, the velocity of convection is not the velocity of the originally coincident fluid particle, but is an average over the volume of heated fluid. This effective velocity of convection will tend to be smaller in magnitude than that of the originally coincident fluid particle, and the larger the volume of heated fluid, the smaller the effective velocity of convection; simply because the correlation between the velocity at two points is a decreasing function of the separation. Thus the greater the molecular diffusivity, the smaller is the effective velocity with which the heat is convected away from the origin and the less the dispersion relative to the origin.

Because the equation governing the diffusion is linear, a wake can be regarded as the superposition of heat spots. Thus the expression (10) will hold for the mean width of the wake at a fixed distance behind the source. This mean width is not to be confused

with the mean instantaneous width which is given by (7). The latter is the mean width of the wake in one particular realization of the turbulence, being measured in principle by taking the average of instantaneous traverses of the wake. The former is essentially the width of the region in which wake fluid is to be found in different realizations; it is determined by time averages at fixed points on a cross-section of the wake and is in fact the quantity actually measured. (The expression (9) for the position of the centroid of a heat spot does not hold for the position of the centroid of the instantaneous wake, because the heat on any cross-section will not all have been released at the same time. The mean-square displacement of the centroid of the instantaneous wake is given by subtracting (7) from (10).)

Although the mathematical results derived from the heat spot analysis are for small times only, the physical arguments would appear to be valid generally. We have to distinguish, however, between the question of the volume of heated fluid and the two-particle diffusion problem. For the latter, it is usually accepted that after a sufficiently large time two particles will diffuse independently and their separation is determined by the solution of the one-particle problem. The overall size of the cloud, as opposed to the actual volume occupied by heated fluid, is determined by the two-particle analysis, and therefore when the cloud is large the rate of increase of its overall size is decelerated by the interaction.

3. — The material auto-correlation coefficient

For diffusion in a field of stationary homogeneous turbulence (or in a situation where the velocity field is statistically homogeneous in the direction of diffusion, such as longitudinal diffusion in turbulent flow through a pipe), the dispersion of a fluid particle is given by

$$\langle Y^2(t) \rangle = 2u^2 \int_0^t (t-\tau)\, S(\tau)\, d\tau\,, \tag{11}$$

where $S(\tau) = \dfrac{\langle u(t)\, u(t+\tau) \rangle}{u^2}$ is the Lagrangian auto-correlation coefficient of the velocity of a fluid particle after a time interval τ. It is also to be noted that $S(\tau)$ is given by

$$u^2 S(\tau) = \int \langle u(\mathbf{x}, \tau)\, u(o, o)\, \theta(\mathbf{x}, \tau) \rangle\, dV \tag{12}*$$

where

$$\frac{D\theta}{Dt} = 0, \quad \theta(\mathbf{x}, o) = \delta^3(\mathbf{x}). \tag{13}$$

It has been shown [5] by considering the motion of an actual molecule that a similar result holds when molecular diffusion is present. In this case, the dispersion of an element of material is given by

$$\langle X^2(t) \rangle = 2\kappa t + 2u^2 \int_0^t (t-\tau)\, S_\kappa(\tau)\, d\tau\,, \tag{14}$$

* This expression is for stationary homogeneous turbulence. It will hold for longitudinal diffusion in a pipe if the brackets also denotes an average over all possible positions of the origin across the cross-section of the pipe, as well as over the ensemble of all realizations.

where $S_\kappa(\tau)$ is called the material auto-correlation coefficient. It is the covariance of the velocity of the fluid at the points occupied by an actual molecule of the diffusing material after a time interval τ, the average being over the turbulent ensemble and also over the ensemble of Brownian motions of an actual molecule in each realization of the turbulence. Further, $S_\kappa(\tau)$ is also given by (12) with $\theta(\mathbf{x}, \tau)$ replaced by $\theta_\kappa(\mathbf{x}, \tau)$, the equation for θ_κ being

$$\frac{D\theta_\kappa}{Dt} = \kappa \nabla^2 \theta_\kappa , \quad \theta_\kappa(\mathbf{x}, o) = \delta^3(\mathbf{x}) . \tag{15}$$

Just as $S(\tau)$ represents the persistence of the convection velocity of a fluid particle, so $S_\kappa(\tau)$ represents the persistence of the convection velocity of the material. The interaction of the turbulence and the molecular diffusion is expressed by the difference between S and S_κ. For small values of τ, for which (15) can be solved approximately, it is shown in [6] that

$$S_\kappa(\tau) = S(\tau) - \frac{1}{3} \frac{\kappa w^2 \tau}{u^2} + \kappa O(\tau^2) \tag{16}$$

from which the result (10) follows.

The expression (16) is an alternative formulation of the basic physical idea that molecular diffusion decreases the effective convection velocity away from the origin. The assumption that this effects always persists implies that $S(\tau) < S_\kappa(\tau)$ for all τ, so that

$$\langle X^2(t) \rangle < \langle Y^2(t) \rangle + 2\kappa t .$$

The expressions for the interaction which were derived from the heat spot analysis are of limited value since they are theoretically valid for small times only. It is to be noted however, that TOWNSEND [4] has verified experimentally that the rate of cooling of heat spots is accelerated and moreover that the expression (5) holds quite well (when modified to take account of the decay) for heat spots in decaying grid-generated turbulence for values of the time larger than were expected (a result, incidentally, of some significance for the structure of the turbulence).

For one-particle diffusion, the concept of the material auto-correlation enables an intuitive estimate of the interaction for large times to be obtained. We have

$$u^2 (S(\tau) - S_\kappa(\tau)) = \int \langle u(\mathbf{x}, \tau)\, u(o, o)\, [\theta(\mathbf{x}, \tau) - \theta_\kappa(\mathbf{x}, \tau)] \rangle \, dV. \tag{17}$$

The right-hand-side of (17) is the difference in the convection velocity at time τ of an element of the diffusing material and the originally coincident fluid particle, correlated with the initial velocity. If the Prandtl number ν/κ (ν is kinematic viscosity) is not small compared with unity, the spreading of the diffusing element of material is determined by κ and the velocity shear which produces the differential convection is determined by $\omega = (w^2)^{1/2}$, provided the Reynolds number of the turbulence is not too small. If we assume that (17) is also proportional to $S(\tau)$, this factor being required to take account of the decay of the velocity correlation with time, we have on dimensional grounds

$$u^2 (S(\tau) - S_\kappa(\tau)) = \kappa\omega\, S(\tau)\, f(\omega\tau) , \tag{18}$$

where the underlying physical idea implies that f is an increasing function of $\omega\tau$.

The expression (18) is in agreement with (16) if $f = \frac{1}{3}\omega\tau$ for $\omega\tau \ll 1$. As $\omega\tau$ becomes large compared withh unity, f must tend to a constant α, say (it cannot tend to infinity as this would make $\langle X^2 \rangle$ negative). On substituting into (14), we find that for $\omega t \gg 1$,

$$\langle X^2(t) \rangle = \langle Y^2(t) \rangle - \frac{\alpha\kappa\omega}{u^2} \langle Y^2(t) \rangle + 2\kappa t. \tag{19}$$

We note that $\kappa\omega/u^2 = \sqrt{15}\,(\kappa/\nu)/R_\lambda$, where R_λ is the Reynolds number based on the dissipation length parameter $\lambda = (15\,u^2/\omega^2)^{1/2}$. To see the relative magnitude of the various terms, we use the result that $\langle Y^2(t) \rangle \sim C u^2 t_e t$ for large t, where t_e is the time scale of the energy containing eddies and C is a dimensionless constant of order unity. (It is sometimes convenient to define t_e by $\nu\omega^2 t_e/u^2 = \frac{3}{2}$; then for the decaying grid-generated turbulence in a wind tunnel, $t_e = t_0$ where t_0 is the time from the virtual creation of the turbulence.) Then $\langle Y^2(t) \rangle \sim \frac{1}{10} C\nu R_\lambda^2 t$, and we have

$$\langle Y^2(t) \rangle \sim -C\nu R_\lambda^2 t - \frac{\sqrt{15}}{10}\alpha C\kappa R_\lambda t + 2\kappa t. \tag{20}$$

For sufficiently large Reynolds numbers, it is clear that the $2\kappa t$ term is negligible compared with the decelerated diffusion term, which is itself small compared with the turbulent diffusion term.

The expressions (19) and (20) will break down if the Prandtl number is very small (e. g. liquid metals). The step from (17) to (18) implicitly assumed that the turbulence was responsible for most of the diffusion and that θ and θ_κ were not very different. These assumptions require that the Péclet number (Reynolds number times Prandtl number) is large compared with unity, and they may be all right, so that (19) is valid, even if the Prandtl number is small provided the Péclet number is still large. On the other hand, when the Prandtl number is small, the value of the viscosity, or equivalently the vorticity, should be unimportant; and provided the Péclet number is large, the difference between θ and θ_κ should be controlled mainly by the range of eddies in the inertial subrange. This supposition leads on dimensional grounds to an expression, for $\omega t \gg 1$ and $t \gg (\kappa/\varepsilon)^{1/2}$,

$$\langle X^2(t) \rangle = \langle Y^2(t) \rangle - \alpha^* \frac{(\kappa\varepsilon)^{1/2}}{u^2} \langle Y^2(t) \rangle + 2\kappa t \tag{21}$$

where ε is the rate of energy dissipation per unit mass and α^* is another constant of order unity. Note that $(\kappa\varepsilon)^{1/2}/u^2 = \sqrt{15}\,(\kappa/\nu)^{1/2}/R_\lambda$, so that the difference between (19) and (21) lies only in the different dependence on the Prandtl number in the interaction term.

For small values of the Péclet number, equation (15) for θ_κ can be solved as a power series in the Péclet number, and substitution into (12) then gives S_κ in terms of weighted integrals of the two-time Eulerian correlation. The greater the molecular diffusivity, the smaller the weighting functions will be. The first approximation for zero Péclet number is

$$u^2 S_\kappa(\tau) = (4\pi\kappa\tau)^{-\frac{3}{2}} \int \langle u(\mathbf{x}, \tau)\, u(0, 0) \rangle\, e^{-x^2/4\kappa\tau}\, dV, \tag{22}$$

which suggests as an approximation

$$S_\kappa(\tau) = S(\tau)\, g(\kappa\tau/l^2) \tag{23}$$

where l is the length scale of the energy containing eddies, and $g \to 0$ as $\kappa \to \infty$ and $g(o) = 1$. An expression similar to this with an exponentially decaying function for g was suggested by BURGERS and HINZE (see [6]), but without restriction to small Péclet number. By taking different forms for S and g, various expressions for $\langle X^2(t) \rangle$ can be obtained from (14), but this topic is not worth pursuing further.

The experimental evidence on the effect of the molecular diffusivity is sparse, most experiments having been carried out under conditions where the effect is small. MICKELSEN [7] has measured the spreading of He and CO_2 (Prandtl numbers of order unity) behind a source in grid-generated wind tunnel turbulence. The experimental points are limited but they are reasonably consistent with (19) with $\alpha = 0.23$. In other experiments [8] with H_2 and CO_2 in a turbulent air stream through a pipe, the scatter is sufficient to prevent reliable comparisons with theory being made, but in so far as any conclusion can be made, it is that there is no interaction. In short, the position is that further experiments are required, and these should be carried out at as low a Reynolds number as possible, in order that the effect of molecular diffusion should not be masked by the turbulent diffusion.

4. — The eddy diffusivity

In the absence of molecular diffusion, the flux of material in a statistically steady situation is $\langle u\theta \rangle = K \dfrac{\partial \langle \theta \rangle}{\partial x}$, where K is the eddy diffusivity (this expression is no more than a definition of). When molecular diffusion is present, the flux is

$$\langle u\theta_\kappa \rangle + \kappa \frac{\partial \langle \theta \rangle}{\partial x} = K_\kappa \frac{\partial \langle \theta \rangle}{\partial x}.$$

The difference between the quantities $\langle u\theta \rangle$ and $\langle u\theta_\kappa \rangle$ arises from the different distributions of θ and θ_κ, that of θ_κ being more uniform and less " spotty ". The velocity of convection of an element of diffusing material is not, so to speak, the velocity at a point, but rather the velocity averaged over a small region, the size of which increases so that the effective convection velocity decreases as the molecular diffusivity increases. Thus we expect that $K_\kappa < K + \kappa$, so that the interaction acts to decrease the eddy diffusivity. Alternatively, this may be considered (see [6]) in terms of mixing length theory. An element of diffusing material mixes more rapidly with its surroundings because of molecular diffusion, so that the mixing length and consequently K_κ decrease as K increases.

So far there is no satisfactory theory of this phenomenon, although attempts have been made based on the mixing length theory. The following is a crude attempt at estimating $\langle u\theta_\kappa \rangle$ which in principle would seem to be a sounder approach.

As pointed out by BATCHELOR [2], the spectrum of θ_κ fluctuations is cut off at a wave number $n = \left(\dfrac{\omega}{\kappa}\right)^{1/2}$, provided the Prandtl number is not small compared with unity. Thus the effective convection velocity of the fluctuations of θ_κ should be the velocity

averaged over a length scale n^{-1}. Thus the mean-square effective convection velocity is of order

$$\left(u + n^{-1}\frac{\partial u}{\partial x} + \frac{1}{2}n^{-2}\frac{\partial^2 u}{\partial x^2}\right)^2 = u^2 - \mathrm{O}(n^{-2}\omega^2) - u^2 - \mathrm{O}(\kappa\omega)$$

where u^2 is the local mean-square (component of) velocity. This suggests that

$$\mathrm{K}_\kappa = \mathrm{K}\left(1 - \frac{\beta\kappa\omega}{u^2}\right) + \kappa,$$

which is analogous to (19), where β is a constant of order unity.

The form of K^κ when the Prandtl number is small sompared with unity is not clear. However, by analogy with (21) and (23), the forms of K_κ may be;

$$\mathrm{K}_\kappa = \mathrm{K}\left(1 - \frac{\beta^*(\kappa\varepsilon)^{1/2}}{u^2}\right) + \kappa$$

for small Prandtl number but large Péclet number; and

$$\mathrm{K}_\kappa = \mathrm{K}\,g\left(\frac{\kappa t_e}{l^2}\right) + \kappa$$

for Prandtl and Péclet numbers both small where $g(z)$ is some function which tends to zero as $z \to \infty$.

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DISPERSION IN TURBULENT SHEAR-FLOW

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SOMMAIRE

La théorie de la dispersion de particules de fluide individualisées issues d'un point donné, en turbulence homogène et stationnaire et dans un courant de vitesse moyenne uniforme, est bien établie (Analyse pour une particule).

L'allure asymptotique de la dispersion dans le cas de temps de diffusion importants peut être décrite grâce aux équations de diffusion de Fick, avec un coefficient de diffusion constant, puisque les propriétés statistiques du mouvement des particules fluides sont des fonctions stationnaires du temps.

La distribution de probabilité du déplacement des particules est gaussienne.

Si cette distribution normale s'applique aussi à des temps de diffusion courts, la description de la dispersion avec un coefficient de diffusion, alors variable avec le temps, est également possible.

BATHELOR a montré (Réf. 6) qu'une étude analogue de la dépendance asymptotique dans le temps de la dispersion peut être menée, pour certains types d'écoulements turbulents, stationnaires cisaillés, lorsqu'ils offrent une analogie de structure au long de l'écoulement vers l'aval. Dans le cas d'un écoulement dans un tube ou un canal, les propriétés statistiques des particules de fluide sont des fonctions stationnaires du temps, et l'analyse précédente s'applique directement. La preuve expérimentale est acquise quant à la dispersion longitudinale d'une quantité finie de solution salée dans l'écoulement turbulent de l'eau dans un tube (Réf. 7). Dans l'écoulement libre turbulent cisaillé, présentant une similitude de structure au long de l'écoulement vers l'aval d'une origine donnée, les propriétés statistiques peuvent être considérées comme des fonctions stationnaires d'une nouvelle variable par un choix judicieux des échelles de vitesse et de temps.

Dans le cas hypothétique d'un écoulement cisaillé plan, illimité, et homogène, la diffusion latérale des particules individualisées, à partir d'un point d'émission peut être décrite mathématiquement en retenant l'hypothèse d'un coefficient de diffusion constant. Cependant, d'un point de vue physique, l'existence d'un tel coefficient de diffusion constant est discutable, puisqu'il n'existe pas d'échelle de longueur finie caractérisant l'écoulement.

L'utilisation d'un coefficient de diffusion turbulente est aussi discutable, malgré un coefficient non constant variable avec la position, si l'écoulement cisaillé est inhomogène. Les expériences, relatives à ce cas, sont peu nombreuses, et elles concernent toutes la dispersion latérale à des distances très courtes à l'aval d'une source d'émission fixe. La distribution latérale des particules marquées paraît être oblique.

Deux hypothèses et théories expliquant cette distribution sont discutées.

SUMMARY

The theory of dispersion of marked fluid particles released from a given point in a homogeneous steady turbulence with uniform mean-velocity is well established (one-particle

analysis). The asymptotic behaviour of dispersion for long diffusion times can be described with the Fickian diffusion equation with a constant coefficient of diffusion, since the statistical properties of the fluid particles motion are stationary functions of time. The probability distribution of the particle displacements is Gaussian. If this normal distribution also applies to short diffusion times the description of dispersion with a, now time-dependent, coefficient of diffusion is equally possible.

BATCHELOR (ref. 6) has shown that a similar analysis of the asymptotic dependence of dispersion on time can be made for certain types of steady, turbulent shear flows when they show a similarity in structure in downstream direction. In the case of flow through a tube or channel the statistical properties of fluid particles are stationary functions of time, and the above analysis immediately applies. Experimental evidence is available on the longitudinal dispersion of a finite amount of salt solution in the turbulent flow of water in a pipe (ref. 7). In free turbulent shear flows with a similarity in structure downstream from some origin the statistical properties can be made stationary functions of a new variable by a suitable choice of a velocity and time scale.

In the, hypothetical, case of a plane, unbounded and homogeneous shear flow the lateral diffusion of marked particules from a fixed point source can be described mathematically on the assumption of a constant coefficient of diffusion. However, from a physical point of view the existence of such a constant coefficient of diffusion is questionable, since there is no finite length scale characterizing the flow.

Also questionable is the use of a coefficient of turbulent diffusion, albeit a non-constant one varying with position, if the shear flow is inhomogeneous. Experiments, referring to this case, are few and they all pertain to the lateral dispersion at very short distances downstream from a fixed line source. The lateral distribution of marked particles appears to be skew. Two hypotheses and theories explaining this distribution are discussed.

1. — Introduction

The dispersion of a transferable property by turbulent motion in a shear flow has been the object of many investigations. Two ways of approach have been followed. The first one is based on the assumed analogy between molecular and turbulent motion, both being a random process. The method adopted here is a more formal one, in that just as in molecular diffusionprocesses a coefficient of diffusion is introduced; the problem is reduced to finding a solution of the convective diffusion equation subject to given initial and boundary conditions. The coefficient of eddy diffusion may be assumed to be constant (Boussinesq) or a scalar quantity which may still be some assumed function of place, or connected in one or another way with the turbulent flow pattern (e. g. mixing length theories).

The second way of approach is to consider the dispersion as a random diffusion process of fluid particles as determined by the structure of the turbulence. The average history is studied by following a Lagrangian description of the random walk of the fluid particles.

In principle the two methods are applicable to any type of turbulent flow. With the first method there may be difficulties, even insuperable, of a mathematical nature. But this method may also lead to non-realistic situations, if the assumptions concerning the behaviour of the coefficient of eddy-diffusion are wrong, or even if the description by means of a coefficient of eddy diffusion may be unjustified on physical grounds. A typical example is the dispersion from a fixed source in a plane homogeneous shear-flow with

constant gradient of the mean-velocity. This problem has been solved by LAUWERIER [1], who assumed a constant value of the coefficient of diffusion in the convective diffusion equation. Though the solution for the dispersion downstream of the source is according to what may be expected, namely a skewed lateral distribution, with a greater spread at the side of the smaller value of the mean velocity, yet the situation is untenable on physical grounds. Not only an unbounded homogeneous shearflow with constant gradient can not be realized, but also the existence of a finite, and constant coefficient of eddy diffusion is very questionable since there is neither a finite length scale (integral scale), nor a finite velocity scale, that characterize the turbulent flow pattern.

The second (Lagrangian) approach has been applied with success to dispersion processes in a homogeneous isotropic turbulence. Since the particle velocities are stationary functions of time, and according to the central limit theorem for long diffusion times the probability distribution of the particle displacements is Gaussian, the asymptotic behaviour of dispersion can be described with the Fickian diffusion equation with a constant coefficient of diffusion. So in this case the two approaches become identical. If the normal distribution of particle displacements also applies to short diffusion times the description of dispersion with a now time-dependent, coefficient of diffusion in the Fickian equation appears possible.

The two approaches may not become identical when the turbulent flow is not homogeneous. The turbulence velocities of fluid particles then are no longer stationary functions of time. The relatively simple method of solution obtained for the homogeneous, isotropic, case is no longer applicable. As a matter of fact, only for a few restricted cases partial solutions of the problem of dispersion in flows different from the homogeneous, isotropic, flow have been obtained. These cases shall be considered in this paper. So only the second approach will be considered and no review of results obtained according to the first approach will be given, since it is believed that the first approach does not basically contribute to the knowledge of dispersion by turbulence as such.

2. — Lagrangian description of dispersion

Without too much loss of generality we may confine ourselves to plane, steady turbulent flows, with components of the Eulerian mean-velocity $\overline{U}$ and $\overline{V}$ in the x and y-direction :

$$\overline{U} = \overline{U}(x, y); \overline{V} = \overline{V}(x, y); \overline{W} = 0. \tag{1}$$

The turbulence flow pattern is not homogeneous in either direction.

Consider a marked fluid particle starting at time t_0 from a point (x_0, y) of the flow field. At any time the velocity of the fluid particle is equal to the Eulerian velocity at the point that is passed by the fluid particle at the instant of passing. So

$$U_p(t; t_0, x_0, y_0) = \overline{U}(x, y) + u(t, x, y), \tag{2a}$$

$$V_p(t; t_0, x_0, y_0) = \overline{V}(x, y) + v(t, x, y), \tag{2b}$$

$\overline{U}$, $\overline{V}$ and u, v are the Eulerian mean and turbulence velocity components at the point (x, y) passed by the fluid particle at time t. At this instant the displacement of the fluid particle from the initial position (x_0, y_0) is given by the components X and Y. Hence

$$x(t) = x_0 + X(t); y(t) = y_0 + Y(t)$$

The values of X and Y are given by

$$\mathrm{X}(t; t_0, x_0, y_0) = \int_{t_0}^{t} dt' \, \mathrm{U}_p(t'; t_0, x_0, x_0). \tag{3a}$$

$$\mathrm{Y}(t; t_0, x_0, y_0) = \int_{t_0}^{t} dt' \, \mathrm{V}_p(t'; t_0, x_0, y_0). \tag{3b}$$

The particle velocities U_p and V_p are not stationary random functions of time, since in the expressions (2*a*) and (2*b*) $\overline{\mathrm{U}}(x, y)$ and $\overline{\mathrm{V}}(x, y)$ vary with time because the coordinates x and y of the particle vary randomly with time.

Consider a large number of realizations of the flow and take an ensemble average of particles with the same initial position (x, y). Since the flow is assumed to be steady, the average displacement does not depend on the time t_0, but only on the initial position (x_0, y_0) and the elapse of time $t - t_0$. The initial time t_0 becomes irrelevant, and may conveniently be put equal to zero. Thus we obtain for the average displacement at time t in the x and y direction

$$\overline{\mathrm{X}}(t; x_0, y_0) = \int_{t_0}^{t} dt' \, \overline{\mathrm{U}}_p(t'; x_0, y_0). \tag{4a}$$

$$\overline{\mathrm{Y}}(t; x_0, y_0) = \int_{t_0}^{t} dt' \, \overline{\mathrm{V}}_p(t'; x_0, y_0). \tag{4b}$$

$\overline{\mathrm{U}}_p$ and $\overline{\mathrm{V}}_p$ are the ensemble average values of the Lagrangian velocity components, and still functions of time.

For studying the dispersion by turbulence we may consider the displacement of the fluid particles relative to the average value. The dispersion in the x-direction is

$$\mathrm{D}_x^2(t; x_0, y_0) = \overline{[\mathrm{X}(t; t_0, x_0, y_0) - \overline{\mathrm{X}}(t; x_0, y_0)]^2}$$

Now differentiate first this expression with respect to time

$$\frac{d}{dt}\mathrm{D}_x^2 = 2\,\overline{[\mathrm{X}(t; t_0, x_0, y_0) - \overline{\mathrm{X}}(t; x_0, y_0)]\,[\mathrm{U}_p(t; t_0, x_0, y_0) - \overline{\mathrm{U}}_p(t; x_0, y_0)]}$$

$$= 2\int_0^t dt' \, \overline{[\mathrm{U}_p(t'; t_0, x_0, y_0) - \overline{\mathrm{U}}_p(t'; x_0, y_0)]\,[\mathrm{U}_p(t; t_0, x_0, y_0) - \overline{\mathrm{U}}_p(t; x_0, y_0)]}$$

Integration yields

$$\mathrm{D}_x^2(t, x_0, y_0) = 2\int_0^t dt' \int_0^t dt'' \, \overline{[\mathrm{U}_p(t'; t_0, x_0, y_0) - \overline{\mathrm{U}}_p(t'; x_0, y_0)][\mathrm{U}_p(t''; t_0, x_0, y_0) - \overline{\mathrm{U}}_p(t''; x_0, y_0)]} \tag{5a}$$

Similarly for the dispersion in the y-direction

$$\mathrm{D}_y^2(t; x_0, y_0) = 2\int_0^t dt' \int_0^{t'} dt'' \, \overline{[\mathrm{V}_p(t'; t_0, x_0, y_0) - \overline{\mathrm{V}}_p(t'; x_0, y_0)]\,[\mathrm{V}_p(t''; t_0, x_0, y_0) - \overline{\mathrm{V}}_p(t''; x_0, y_0)]} \tag{5b}$$

The Lagrangian velocity correlations and the dispersion depend not only on (t', t''), but also on the initial position (x_0, y_0) of the marked fluid particles.

3. — Dispersion in homogeneous turbulence with constant mean-velocity

If the turbulent flow is homogeneous and with a constant meanvelocity (in the x-direction, say), then the velocities of the fluid particles are stationary functions of time. Because of the homogeneity of the flow the expressions (5*a*) and (5*b*) for the dispersion become independent of the initial position of the fluid particle, and the Lagrangian correlations are symmetric functions of the time.

This case has been studied first by Sir Geoffrey TAILOR [2] and after him by many others (see e. g. references [3] and [4]) :

Since $\overline{U} =$ constant, $\overline{V} = 0$ we obtain from (2*a*) and (2*b*)

$$\overline{U}_p = \overline{U} \quad \text{and} \quad \overline{V}_p = 0$$

From (4*a*) and (4*b*) we obtain for the average displacement

$$\overline{X}(t) = \overline{U}t \; ; \quad \overline{Y} = 0$$

Introduce the Lagrangian correlation coefficients

$$_uR_L(\tau) = \frac{\overline{u(t)\, u(t-\tau)}}{u'^2} \; ; \quad _vR_L(\tau) = \frac{\overline{v(t)\, v(t-\tau)}}{v'^2} \tag{6}$$

where $u' = \sqrt{u^2}$, $v' = \sqrt{v^2}$, and $t' - t'' = \tau$.

The expressions (5*a*) and (5*b*) then reduce to the well known forms

$$D_x^2(t) = \overline{(X - \overline{X})^2} = 2u'^2 \int_0^t d\tau\, (t-\tau)\, _uR_L(\tau) \tag{7a}$$

and

$$D_y^2(t) = \overline{Y^2} = 2v'^2 \int_0^t d\tau\, (t-\tau)\, _vR_L(\tau) \tag{7b}$$

An assumption, confirmed by experimental evidence, is that the Lagrangian correlations become zero at large values of the argument. So for large values of the time t we may write

$$D_x^2(t) = 2u'^2 \left[t \int_0^\infty d\tau\, _uR_L(\tau) - \int_0^\infty d\tau\, \tau\, _uR_L(\tau) \right] \to 2u'^2\, t \int_0^\infty d\tau\, _uR_L(\tau) \tag{8a}$$

and

$$D_y^2(t) = 2v'^2 \left[t \int_0^\infty d\tau\, _vR_L(\tau) - \int_0^\infty d\tau\, \tau\, _vR_L(\tau) \right] \to 2v'^2\, t \int_0^\infty d\tau\, _vR_L(\tau) \tag{8b}$$

Since according to the central limit theorem the probability distribution of X and Y becomes Gaussian as $t \to \infty$, a coefficient of eddy diffusion may be introduced

$$\epsilon_x = \frac{D_x^2}{2t} = u'^2 \int_0^\infty d\tau\, _uR_L(\tau) \tag{9a}$$

$$\epsilon_y = \frac{D_y^2}{2t} = v'^2 \int_0^\infty d\tau\, _vR_L(\tau) \tag{9b}$$

The dispersion can be described with the Fickian diffusion equation with the above constant, diffusion coefficient.

BATCHELOR [5] has shown that if the probability distribution of X and Y is Gaussian at all times the dispersion can still be described with the Fickian diffusion equation, if the coefficient of eddy diffusion is given by

$$\epsilon = \frac{1}{2}\frac{d}{dt}\mathrm{D}^2 \tag{10}$$

Hence from (7*a*) and (7*b*) we obtain

$$\epsilon_x = \frac{1}{2}\frac{d}{dt}\mathrm{D}_x = u'^2 \int_0^t d\tau \, {}_u\mathrm{R}_\mathrm{L}(\tau) \tag{11a}$$

and

$$\epsilon_y = \frac{1}{2}\frac{d}{dt}\mathrm{D}_y = v'^2 \int_0^t d\tau \, {}_v\mathrm{R}_\mathrm{L}(\tau) \tag{11b}$$

The above results have been confirmed satisfactorily well by many experiments on the diffusion of heat from a fixed source placed in a grid-produced turbulence, taking into account the limited accuracy of this kind of experiments.

4. — Dispersion in non-homogeneous, self-preserving, turbulent flows

The grid-produced turbulence mentioned in the previous section, is strictly not homogeneous even at large distances downstream of the grid because of the decay of the turbulence in downstream direction. Yet the experimental results on the diffusion of heat from a fixed source are not in contradiction with theoretical results based upon the assumed homogeneity of the flow and upon the assumption that the velocities are stationary randomfunctions of time. So for long diffusion times the dispersion D_y appears to increase linearly with time. The explanaiton is that these experiments have been carried out in a region of the flowfield where the turbulence during decay exhibits a large degree of selfpreservation, i. e. the turbulence structure remains almost similar at different values of the time or distance to the grid. In that case the characteristic length scale of the turbulence varies as $t^{1/2}$, while the characteristic velocity scale of the turbulence varies as $t^{-1/2}$, so that the time scale of the turbulence processes varies proportional to t. Hence it is not the turbulence velocity $v(t)$ that appears to be a stationary random function of time but $t^{1/2} \cdot v(t)$; so $v'(t) \cdot t^{1/2}$ is a constant. BATCHELOR and TOWSEND [3] suggest to take a new variable t^* defined by

$$dt^* = \frac{dt}{t} \quad \text{or} \quad t^* = ln\frac{t}{t_1} \tag{12}$$

where t_1 is arbitrary, for describing the dispersion process in a decaying turbulence.

BATCHELOR [6] has shown that this procedure may be applied to other types of flows where the turbulence is self preserving (namely in free turbulent shear flows as jets and wakeflows) in order to obtain relations for the asymptotic behaviour of dispersion processes for long diffusion times.

Before discussing further the dispersion in free turbulent shear flow, brief consideration will be given of the axial dispersion in a steady turbulent flow through a straight conduit of uniform cross section. Here the flow and its average statistical properties are homogeneous in the flow direction, so that the velocities of the fluid particles become

stationary random functions of time after a sufficiently long time after their release, notwithstanding the fact that the particles move through different parts of a cross section with different values of the mean velocity (ref. 6). Then the influence of the initial position of the particle at $t = t_0$ becomes diminishingly small, and the axial displacement of the particle, in a single realization of the flow becomes

$$\overline{\mathrm{X}}(t) \simeq \int_0^t dt' \, \overline{\mathrm{U}}_p(t') \simeq \overline{\mathrm{U}}_{av} \, t \tag{13}$$

where $\overline{\mathrm{U}}_{av}$ is the average velocity in a cross section of the conduit. This relation becomes more correct as the time increases. If the probability distribution of finding a particle at a point of a cross section is uniform over a cross section, in one single realization of the flow after a very long time a fluid particle, moving freely over the cross section as it flows downstream, attains all values of the mean velocity and the result is an average value equal to the discharge velocity of the flow. So all fluid particles must have the same mean velocity equal to the average, discharge velocity. This result has been confirmed by experiments by Sir Geoffrey TAYLOR [7] with injected salt in a flow through a straight circular pipe; the velocity of the position of the maximum concentration of salt was equal to $\overline{\mathrm{U}}_{av}$. Also the results of experiments with a suspension of small spheres in water (BATCHELOR, BINNIE and PHILLIPS [13]) have shown that the average residence time of a large number of particles during the flow through a given length of the pipe was equal to the length of the pipe divided by the discharge-velocity. The small difference observed could be accounted for by the effect of the finite size of the particles.

In the case of free turbulent shear flows as jets and wakes at a certain distance from a virtual origin the structure is similar in sections downstream (self preservation), so that this structure can be expressed in terms of a length-scale and a velocity-scale which vary with the distance x to the virtual origin. Let $\mathrm{L}(x)$ and $V(x)$ be the Eulerian length and velocity scale respectively. The corresponding time scale $\mathrm{T}(x) = \mathrm{L}/V$. For a particle the statistical properties of its velocity $\mathrm{U}_p(t)$ vary with the distance x. Let $v(t)$ and $\mathcal{T}(t)$ be the velocity and time scale respectively of the particle motion. It is reasonable to assume that at a mean distance travelled after time t by particles, released at the virtual origin, equal to $\overline{\mathrm{X}}(t)$, the velocity scal $v(t) = V\ [\overline{\mathrm{X}}(t)]$ and the time scale $\mathcal{T}(t) = \mathrm{T}[\overline{\mathrm{X}}(t)]$.

Now if the turbulence preserves its structure as it changes in downstream direction, then the particle velocity $\mathrm{U}_p(t)$ may be made a stationary random function of a new variable t^* by a suitable choice of the length and velocity scales, and hence of the time scale. Thus BATCHELOR suggests

$$dt^* \propto \frac{dt}{\mathcal{T}(t)} \tag{14}$$

and

$$\frac{\mathrm{U}_p(t) - \mathrm{U}_0}{v(t)} = \mathrm{F}(t^*) \tag{15}$$

where $\mathrm{F}(t^*)$ is a stationary function of t^*. The constant velocity U_0 is not zero in the case of wake flows, otherwise it is usually zero.

BATCHELOR put $\mathrm{L} \propto x^p$, $V \propto x^{-q}$, so that

$$v(t) \propto [\overline{\mathrm{X}}(t)]^{-q} \quad \text{and} \quad \mathcal{T}(t) \propto [\overline{\mathrm{X}}(t)]^{p+q}.$$

Making use of the relevant values for p and q (e. g. for jetflows $p=1$, wakeflows $p+q=1$) it then turns out that for all types of flow considered $\mathcal{F}(t) \propto t$ and consequently again

$$t^* = ln \frac{t}{t_1}$$

Thus for all these flows, including decaying-isotropic turbulence the time scale of the turbulent motions varies proportionally with the time t measured from the instant of release of the particles at the virtual origin.

Furthermore BATCHELOR obtained the important result that the dispersion $D_x(t)$ and $D_y(t)$ are proportional with the width of the shear layer at $\overline{X}(t)$, though the axial dispersion D_x may be much larger than the lateral dispersion D_y. For long diffusion times a coefficient of eddy diffusion may be introduced which for jet type flows varies proportionally with $\overline{X}(t)^{1-q}$ (for a round free jet, $q=1$ and the eddy diffusion coefficient becomes a constant), for wake type flows proportional with $\overline{X}^{1-2q}$ (in a plane wake $q=1/2$, hence the eddy diffusion coefficient is here constant too).

5 .— Dispersion in a homogeneous shear-flow

CORRSIN [8] has considered the dispersion from a point source in an unbounded homogeneous plane parallel isotropic-turbulent flow with a mean velocity in the x-direction and which has a constant gradient $d\overline{U}/dy$.

Such a flow is not physically possible, and therefore the case is purely hypothetical. Assume for simplicity $U=0$ at $y=0$.

From eqs (2a) and (2b) we obtain

$$U_p(t) = \overline{U}(y) + u(x,y) = \frac{d\overline{U}}{dy} Y(t) + u(t) \tag{16a}$$

$$V_p(t) = v(x,y) = v(t) \tag{16b}$$

The displacement of a fluid particle originating from the source follows from (4a) and (4b)

$$X(t) = \int_0^t dt' \left[\frac{d\overline{U}}{dy} Y(t') + u(t') \right] \tag{17a}$$

$$Y(t) = \int_0^t dt'\, v(t') \tag{17b}$$

From these relations follows for the average particle velocity and the average displacement obtained from a large number of realizations

$$\overline{U}_p = \overline{V}_p = 0 \quad \text{and} \quad \overline{X} = \overline{Y} = 0.$$

The dispersion from the point source in the x-direction

$$D_x^2(t) = 2 \int_0^t dt' \int_0^{t'} dt'' \overline{\left[\frac{d\overline{U}}{dy} Y(t'') + u(t'') \right] \left[\frac{d\overline{U}}{dy} Y(t') + u(t') \right]} \tag{18a}$$

Similarly in the y-direction

$$D_y^2(t) = 2\int_0^t dt' \int_0^{t'} dt'' \, \overline{v(t')\, v(t'')} \tag{18b}$$

With the introduction of the Lagrangian coefficients

$${}_u R_L(\tau) = \frac{\overline{u(t)\; u(t-\tau)}}{u'^2} \quad \text{and} \quad {}_v R_L(\tau) = \frac{\overline{v(t)\; v(t-\tau)}}{v'^2},$$

CORRSIN obtained

$$D_x^2(t) = 2u'^2 \int_0^t d\tau \; (t-\tau)\, {}_u R_L(\tau) +$$

$$+\left(\frac{d\overline{U}}{dy}\right) 2\, v'^2 \left[\frac{2}{3} t^3 \int_0^t d\tau \, {}_v R_L(\tau) - t^2 \int_0^t d\tau \, \tau \, {}_v R_L(\tau) + \frac{1}{3} t \int_0^t d\tau \, \tau^2 \, {}_v R_L(\tau)\right. \tag{19a}$$

and

$$D_y^2(t) = 2v'^2 \int_0^t d\tau \, (t-\tau) \, {}_v R_L(\tau) \tag{19b}$$

If $R_L(\tau)$ becomes zero at large values of τ so that $\int_0^\infty d\tau \, R_L(\tau)$ is finite $D_y(t)$ would vary as $t^{1/2}$ and $D_x(t)$ as $t^{3/2}$. However in this type of flow it is not obvious without more that the correlation becomes zero at large distances. On the contrary it might equally well be justified to assume a finite correlation at any value of τ and the asymptotic behaviour discussed above does no longer hold. For the same reason it is not justified to introduce a coefficient of eddy diffusion on the ground of such an asymptotic behaviour.

For very short diffusion times so that the Lagrangian correlation coefficients are approximately equal to one, the expressions (19a) and (19b) reduce to

$$D_x^2(t) \simeq u'^2 t^2 + \frac{5}{18} v'^2 \left(\frac{d\overline{U}}{dy}\right)^2 t^4 \approx u'^2 t^2 \tag{20a}$$

$$D_y^2(t) \simeq v'^2 t^2 \tag{20b}$$

As may be expected the expressions for $t \to 0$ are identical with those for a homogeneous field with zero mean-velocity gradient. However, depending on the mean-velocity gradient in the shear-flow the effect of this gradient on the longitudinal dispersion becomes noticeable with increasing diffusion time.

6. — Dispersion from a fixed source in a shear flow

We may again consider a steady flow parallel to the x axis, and with the $\overline{U}$ component being only a function of the y-coordinate. The flow is homogeneous in the x-direction but not in the lateral direction. Such a flow occurs for instance in a straight channel or in a plane Couette flow. It may also be considered as a first approximation to actual flows that are not strictly parallel flows. As a matter of fact the experiments known on the dispersion from a fixed source, namely those by DRYDEN and SKRAMSTAD [9] in a turbulent boundary layer, by CORRSIN and UBEROI [10] in a round free jet, and by HINZE and VAN DER HEGGE ZIJNEN [11] in a plane free jet, are not

made in a parallel flow, but for the study of the dispersion the flow in the neighbourhood of the fixed source may be considered as almost parallel.

The flows investigated are not homogeneous. From the general considerations given in section 2 it becomes clear that an interpretation of experimental results (in particular when the diffusion times are long) in the light of theories on the statistical behaviour of fluid particles in an homogeneous turbulence is difficult, if not impossible, to make. Now it turns out that all the experiments mentioned above refer to short diffusion times. No data are available on diffusion times long compared with the Lagrangian integral scale, say. The reason for this is that the turbulent flows studied show a high value of the relative intensity of the turbulence, especially in the case of the free turbulent jets. All experimenters used a thin heated wire as a line source of heat. The wire had to be taken thin because it should approach point source conditions and it should not disturb the flow too much. So the heat capacity of the hot wire was very small. The turbulence spread of the heat produced by the wire within the integral scale of turbulence was already so great the at longer diffusion times the temperature rise became hardly measurable with reasonable accuracy. The measurements remained confined to the direct neighbourhood of the hot wire. In this small region the meanvelocity could be assumed to have a constant gradient as shown in eqs (20) was negligibly small. According to these eqs (20) the wire increasing linearly with distance. Apparently the direct effect of the meanvelocity gradient as shown in eqs (20) was negligibly small. According to these eqs (20) the lateral dispersion is given by

$$D_y^2\ (t) = \overline{Y}^2 = v'^2 t^2$$

The observed lateral distribution of the heat was skew, with the greater spread at the side of the greater value of the mean velocity. It may be remarked that this skewness is just opposite to that which would occur if the dispersion could be described by means of a constant coefficient of diffusion.

CORRSIN [12] assumed that the lateral spread at a cross section X downstream of the wire was directly determined by the probability distribution of the v-component of the turbulence velocity. In other words, the lateral temperature distribution is simply the probability density of the lateral velocity fluctuation.

Hence the skewness of the Y-distribution is caused by the skewness of the v-distribution, and the skewness factor S_y is equal to the skewness factor S_v. A skewness of the v-distribution may be expected since the distribution of the relative intensity $\frac{v'}{U}$ is not uniform. Measurements on the distribution of $\frac{v'}{U}$ and on the skewness S_y showed that

$$\frac{d\overline{U}}{dy} \text{ and } \frac{d}{dy}\frac{v'}{U} \text{ have opposite signs}$$

$$\frac{d\overline{U}}{dy} \text{ and } S_y \text{ have the same sign.}$$

With the restriction of small skewness and small deviations from the Gaussian distribution, CORRSIN choose the following probability density distribution P :

$$P\left(\frac{v}{\overline{U}}\right)=\frac{1}{2\pi\left(\frac{v'}{\overline{U}}\right)_0}\exp\left\{\frac{-\left(\frac{v}{\overline{U}}+c\right)^2}{\left[\sqrt{2}\left(\frac{v'}{\overline{U}}\right)_0-\epsilon\right]^2}\right\}\text{ for }-\infty<\frac{v}{\overline{U}}<-c$$

$$=\frac{1}{2\pi\left(\frac{v'}{\overline{U}}\right)_0}\exp\left\{\frac{-\left(\frac{v'}{U}+c\right)^2}{\left[\sqrt{2}\left(\frac{v'}{\overline{U}}\right)_0+\epsilon\right]^2}\right\}\text{ for }-c<\frac{v}{\overline{U}}<+\infty \qquad (21)$$

Since negatively directed velocities originate in the $y>0$ region we may expect the negative fluctuations $\left(\frac{v}{\overline{U}}\right)_-$ to be a measure of the turbulence intensity at $y=+l$, say. Conversely $\left(\frac{v}{\overline{U}}\right)_+$ to be a measure of this intensity at $y=-l$. So to a first approximation, assuming a small value of the turbulence-intensity gradient, we may put

$$\left(\frac{v}{\overline{U}}\right)_-\simeq-\left(\frac{v'}{\overline{U}}\right)_0-\left(\frac{d}{dy}\frac{v'}{\overline{U}}\right)_0 l$$

$$\left(\frac{v}{\overline{U}}\right)_+\simeq+\left(\frac{v'}{\overline{U}}\right)_0-\left(\frac{d}{dy}\frac{v'}{\overline{U}}\right)_0 l$$

The constant c which accounts for the shift of the maximum in the probability-density distribution curve $P\left(\frac{v}{\overline{U}}\right)$ in the negative $\left(\frac{v}{\overline{U}}\right)$ — direction, and the constant ϵ are related to the length l and the gradient of the relative intensity of the turbulence. Satisfying the basic relations for the probability density, there is obtained

$$\epsilon=\frac{1}{2}\sqrt{\pi}\cdot c \quad\text{and}\quad c=-\frac{1}{\frac{1}{2}\sqrt{\frac{\pi}{2}}-\frac{1}{\sqrt{2\pi}}}\cdot l\left(\frac{d}{dy}\frac{v'}{\overline{U}}\right)_0$$

The skewness factors then are approximately

$$S_y=S_v\simeq-\frac{4l}{\left(\frac{v'}{\overline{U}}\right)_0}\left(\frac{d}{dy}\frac{v'}{\overline{U}}\right)_0 \qquad (22)$$

One of the basic relations mentioned above which has to be satisfied by $P\left(\frac{v}{\overline{U}}\right)$ is, that its first moment should be zero. Hence also the first moment of the lateral temperature distribution should be zero; the centroid of this distribution curve should coincide with $y=0$.

Now the measured temperature distribution curves did show a finite value for the first moment $\overline{Y}$. Furthermore in the free jet experiments the gradient of the relative

intensity had a sign opposite to that of the absolute intensity. Thus $\left(\frac{d}{dy}\frac{v'}{\overline{U}}\right)_0$ and $\frac{1}{\overline{U}_0}\left(\frac{dv'}{dy}\right)_0$ had opposite signs.

Since the skewness of the v-component reflects the distributions of the lateral turbulence fluctuations, $\frac{1}{\overline{U}_0}\left(\frac{dv'}{dy}\right)_0$ should determine the skewness rather than $\left(\frac{d}{dy}\frac{v'}{\overline{U}}\right)_0$. Consequently opposite signs of the skewness-factor S_y and S_v may be expected.

HINZE [4] followed a different approach to explain the observed skewness of the temperature distribution curve. He connected this skewness with the local turbulence shear stress $-\rho\,\overline{uv}$. If for instance, $\frac{d\overline{U}}{dy} > 0$ and consequently $\overline{uv} < 0$, on the average more fluid particles with negative u will be transported with a positive v through a control plane; conversely for negative values of v on the average more particles with positive u will pass the control plane. To positive values of v correspond positive values of y. Hence for the same value of X, particles with a positive v (positive Y) need a longer time to reach this value of X than particles with negative v (negative Y). Consequently the particles with positive v may have travelled a longer distance Y before reaching the value of X. For at short diffusion times

$$\begin{aligned} \mathrm{Y}(t) &\simeq v_0\, t \\ \mathrm{X}(t) &\simeq (\overline{U}+u)_0\, t \end{aligned} \tag{23}$$

Hence

$$\mathrm{Y}(t) \equiv \mathrm{Y}(x) = \frac{v_0}{(\overline{U}+u)_0}\cdot \mathrm{X} = \frac{v_0}{(\overline{U}+u)_0}\,x \tag{24}$$

The relation (24) immediately shows that indeed $\overline{\mathrm{Y}}(x) \neq 0$, and that the probability density distribution of Y is not identical with that of v. Only when $\frac{u'}{\overline{U}} \ll 1$, is it the case approximately.

Since $\frac{u}{\overline{U}} < 1$, we may expand (24) in a TAYLOR series. We then obtain for the average values of Y, Y^2 and Y^3 :

$$\frac{\overline{\mathrm{Y}}}{x} = -\frac{(\overline{uv})_0}{\overline{U}_0^2} + \frac{(\overline{u^2 v})_0}{\overline{U}_0^3} - \frac{(\overline{u^3 v})_0}{\overline{U}_0^4} + \cdots \tag{25a}$$

$$\frac{\overline{\mathrm{Y}^2}}{x^2} = \frac{(\overline{v^2})_0}{\overline{U}_0^2} - 2\frac{(\overline{uv^2})_0}{\overline{U}_0} + 3\frac{(\overline{u^2 v^2})_0}{\overline{U}_0} \tag{25b}$$

$$\frac{\overline{\mathrm{Y}^3}}{x^3} = \frac{(\overline{v^3})_0}{\overline{U}_0^3} - 3\frac{(\overline{uv^3})_0}{\overline{U}_0^4} + 6\frac{(\overline{u^2 v^3})_0}{\overline{U}_0^5} \tag{25c}$$

Thus the skewness factor $S = \overline{\mathrm{Y}^3}/(\overline{\mathrm{Y}^2})^{3/2}$ can be expressed in terms of the skewness factor S_v.

According to eq (25*a*) the shift $\overline{Y}$ of the centroid of the temperature distribution is only proportional to the local shear stress if the effect of the higher order correlations may be neglected.

In order to calculate the skewed temperature distribution HINZE assumed a normal joint-probability-density distribution of the turbelence velocity components u and v.

$$F\left(\frac{u}{\overline{U}}, \frac{v}{\overline{U}}\right) = \frac{\overline{U}^2}{2\pi u' v' [1 - R_{uv}^2]^{1/2}} \exp\left\{\frac{-1}{2(1-R_{uv}^2)}\left[\frac{u^2}{u'^2} - \frac{2uvR_{uv}}{u'v'} + \frac{v^2}{u'^2}\right]\right\} \quad (26)$$

where $R_{uv} = \dfrac{(\overline{uv})_0}{u'v'}$.

Consider an element $dxdy$ at the point (x, y). The probability of finding in this element $dxdy$ a fluid particle originating from the source of unit strenght at ($x = 0$, $y = 0$) reads.

$$P(x, y)\, dxdy = \frac{1}{\overline{U}^2} \int_{-\infty}^{t} dt_0\, F\left(\frac{u}{\overline{U}}, \frac{v}{\overline{U}}\right) \cdot dudv$$

With $du = dx/(t - t_0)$, $dv = dy/(t - t_0)$ and the relations (23) and (26) the temperature distribution can be calculated. For the normalized distribution, which is equal to $P(\eta)/P(0)$ there is obtained (see ref. 4)

$$\frac{P(\eta)}{P(0)} = \frac{1}{\left[1 - 2\frac{u'}{v'} R_{uv}\eta + \left(\frac{u'}{v'}\right)^2 \eta^2\right]^{1/2}} \exp\left\{\frac{-\overline{U}^2\eta^2}{2v'^2\left[1 - 2\frac{u'}{v'} R_{uv}\eta + \left(\frac{u'}{v'}\right)^2 \eta^2\right]}\right\} \quad (27)$$

where $\eta = y/x$.

Comparison of this result with the measured temperature distribution in the wake of a hot wire placed in a plane free jet showed satisfactory agreement in the region $-0.2 < \eta < 0.2$. Beyond this region the computed distribution was too skew. In the relation (27) the value of R_{uv} obtained from $\overline{Y}/x$ according to (25*a*) (neglecting the higher order terms) was used, while the values of u' and v' were obtained by trial and error. The skewness factor S_y could be determined from the measured temperature distribution, the skewness factor S_v could be calculated from the relation between S_y and S_v. This relation has been obtained by making use of the joint-probability density distribution $F\left(\frac{u}{\overline{U}}, \frac{v}{\overline{U}}\right)$ and some assumptions suggested by BATCHELOR and TOWSEND [3] concerning third and fourth order correlations :

$$S_y \simeq S_v - 9\frac{u'}{\overline{U}} R_{uv} + 3\, S_v^2 \frac{u'}{\overline{U}} R_{uv} \quad (28)$$

This relation shows that the skewness factors can have opposite signs.

Though within the, rather restricted, accuracy of the measurements and of the method used the agreement was satisfactory, yet it must be kept in mind that the assumption of a normal joint probability density distribution of u and v is probably not justified. The marginal distribution of u and v then are normal, in contradiction with the skewed distribution of v calculated from eq (28). Then the theory is only valid for small diffusion times, that is small values of $x/(\overline{U} + u)_0$ and y/v_0. An estimate shows that for the conditions pertinent to the experiments, the theory may only be applied to values of $\eta < v'/\overline{U}$, which is roughly 0.2 in these experiments.

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DISCUSSION DE LA SECTION : DIFFUSION ET POINT DE VUE DE LAGRANGE

Professeur J. LUMLEY
Professeur P. G. SAFFMAN
Secrétaires scientifiques

Professeur CORRSIN introduces Professor LUMLEY.

Professor LUMLEY (U.S.A.). — « The Mathematical Nature of the Problem of Relating Lagrangian and Eulerian Statistical Functions in Turbulence ».

Professor CORRSIN called for comments and questions.

Dr FRENKIEL (U.S.A.). — Asked in what connection the speaker wished to introduce Markov processes.

Speaker. — Markov processes would perhaps be used to generate test functions.

Dr KRAICHNAN (U.S.A.). — Asked if the speaker would regard a result obtained by the Monte-Carlo method as theoretical or experimental.

Speaker. — The result would be regarded as theoretical since the technique would be used to carry out an integration, rather than using the machine as a model of the physical problem on which experiments could be performed.

Professor YAGLOM (U.S.S.R.). — Wondered in what connection the Feynman integrals had been introduced since it seemed to him that these integrals are only a special notation for the solutions of the Schroedinger equations.

Speaker. — The Feynman integral was introduced simply as an example of a similar problem.

Professor YAGLOM (U.S.S.R.). — Pointed out that the Monte-Carlo method can be used for the evaluation of Wiener integrals and more general functional integrals when the measure in function space corresponds to some Markov process and wondered if the speaker was familiar with the report of a Monte-Carlo evaluation of a Wiener integral in the paper by Gelfand and Chentsov in the Russian Journal of the Theoretical and Experimental Physics. (Vol. 31, p. 1106, 1956).

Speaker. — Said that he has seen the reference to this article in the survey by Brush, « Functional Integrals and Statistical Physics » published in the Reviews of Modern Physics.

Professor S. CORRSIN. — « Theories of Turbulent Dispersion ».

Chairman called for comments and questions.

Professor FAVRE (France). — Called for suggestions for further work to be carried out in the Institute and inquired whether there was continued interest in space-time correlations.

Speaker. — Indicated that strong interest existed.

Professor FAVRE. — Inquired as to whether work should be done in grid turbulence or in a boundary layer.

Speaker. — Indicated that measurements behind a grid would probably be preferable because af the simplicity of the flow.

Professor FAVRE. — Pointed out that measurements in a boundary layer may be more significant due to the non-zero production terms.

Professor LIEPMANN (U.S.A.). — Pointed out that the boundary layer is more complex due to the strong influence of viscosity near the walls.

Professor KOVÁSZNAY (U.S.A.). — Suggested that measurements of scalar fluctuations, i. e., temperature would present the simplest case.

Professor FAVRE (France). — Suggested possibility of making space-time double and triple correlation measurements in a fully developed pipe flow.

Professor STEWART (Canada). — Pointed out that it seemed preferable to design experiments that would be truly comparable to the theory. In this case in any turbulent field it would surely be preferable to measure space-time correlations for single hot spots, catching the same spot at more than one place.

Professor SAFFMAN (Gr. Brit.). — « Some Aspects of the Effects of the Molecular Diffusivity in Turbulent Diffusion ».

Chairman called for comments and questions.

Dr KRAICHNAN (U.S.A.). — Inquired as to the smallest value of R_λ to which the equation

$$\overline{X^2}(\tau) = \overline{Y^2}(\tau)\left[1 - \frac{\sqrt{15}\,\alpha\kappa}{\nu R_\lambda}\right] + 2\kappa\tau$$

might be valid. Did the speaker think that there might be circumstances in which the interaction of turbulent and molecular dispersion decreased with decrease of R_λ ? Is $R_\lambda \cong 1$ the condition in which interaction is most significant ?

Speaker. — The equation quoted was derived by intuitive arguments applied to the expression (17) of the paper, and the arguments implicitly assumed the existence of a Kolmogoroff range of eddy sizes. Until the argument can be made rigorous, it is not possible to specify the smallest value of R_λ for which it may be valid. A necessary, but perhaps not sufficient, condition for the equation to be valid is that the interaction term should be small compared with $\overline{Y^2}(\tau)$.

The interaction of turbulent and molecular diffusion decreases with decrease of the Reynolds number, as shown by the expression (20); it is the ratio of the interaction term to $\overline{Y^2}(\tau)$ which increases as R_λ decreases. I do not know whether this ratio has a maximum. As long as $\overline{X^2}(\tau)$ is positive, the expression is not nonsense.

Professor YAGLOM (U.S.S.R.). — Wished to make a comment related to the paper of Professor Corrsin as well as that of Professor Saffman. He pointed out that the representation of the mean square dispersion in terms of the Lagrangian spectrum is a special case of the representation (Trans. Amer. Math. Soc., v. 50, n° 2, 226-257, 1941) given by Kolmogoroff (Dok. Akad. Nauk SSSR, v. 26, 6-9, 1940) and also by Schoenberg and Von Neumann for a random function with stationary increments. These authors had shown that such a function could be represented as an integral of a velocity only if the second moment of the spectrum converged, but if molecular motion is taken into account the displacement function is not differentiable and the second moment of the spectrum diverges logarithmically. Professor Yaglom also wished to direct attention to the similarity of the work reported by Professor Saffman and that done recently by E. Novikov from the Institute of Atmospheric Physics, Moscow (see Dokl. Akad. Nauk. SSSR, vol. 139, n° 3, 1961). The principal results of Novikov's work relate to the form of the spectrum for very large wave numbers. It was suggested that it might be appropriate to present a summary of this work after the session on Energy Transfer in Homogeneous Turbulence.

Professor LIEPMANN (U.S.A.). — Questioned the non-existence of the derivative referred to by Professor Yaglom. He felt that this implied a contradiction between the molecular and continuum points of view.

Professor YAGLOM (U.S.S.R.). — Suggested that the situation was similar to Brownian motion and that the small scale motion was primarily dependent on molecular agitation.

Professor HINZE (Holland). — « Dispersion in Turbulent Shear Flow ».

Chairman called for comments and questions.

Professor LIEPMANN (U.S.A.). — Inquired whether the central limit theorem could be quantitatively applicable.

Speaker. — Indicated that the application was only qualitative.

Professor LIEPMANN and Professor CORRSIN (U.S.A.). — Pointed out that there exists no formal central limit theorem for continuous dependent variables.

Professor TAYLOR (Gr. Brit.). — Wished to bring to the attention of the group some recent work by Mr Bretherton of Cambridge, England on dispersion in non-turbulent shear flow. He felt that the results were of considerable interest.

Professor LUMLEY (U.S.A.). — Pointed out that the result of Batchelor, Binnie and Phillips mentioned by Professor Hinze could be obtained as a special case of the Liouville-type theorem mentioned in Lumley's talk.

COMMENTAIRE DE LA SECTION : DIFFUSION ET POINT DE VUE DE LAGRANGE

par le Professeur S. CORRSIN
Président

M. le Président, Messieurs, pendant ce Colloque nous avons entendu quatre langues : le bon français, le mauvais français, le bon anglais, le mauvais anglais. Ce matin, vous serez exposés à en entendre une cinquième : l'anglais-français.

Je m'excuse de lire ma conférence, mais autrement vous ne pourriez pas reconnaître du tout qu'elle est vraiment une langue.

Je veux remercier les personnes qui ont aidé à la traduction, particulièrement M. Verollet.

La diffusion à partir d'une source ponctuelle dans un champ de turbulence homogène s'exprime plus aisément dans le système de coordonnées de Lagrange. Mais dans ce système les équations de Navier-Stokes sont très difficiles à utiliser. Donc le problème théorique peut se définir comme la recherche d'une relation entre les fonctions statistiques du système de Lagrange et du système d'Euler. L'analyse dynamique et l'expérience semblent toutes deux plus simples dans le système d'Euler. Notre premier objectif est de prédéterminer la valeur du carré moyen du déplacement d'une particule.

Dans la première conférence, sur « Le caractère mathématique du problème de la relation entre les fonctions statistiques Lagrangiennes et Eulériennes », le Docteur LUMLEY a démontré que les moyennes prises en un seul point dans chacun des deux systèmes sont égales l'une à l'autre si la densité du fluide est constante. C'est un résultat que tout le monde a appliqué intuitivement. Je pense qu'il soulève des questions sérieuses pour les problèmes dans lesquels la densité peut avoir de grandes fluctuations.

Il a traité également de l'établissement d'une relation entre les champs de vitesse Lagrangien et Eulérien. Une telle relation n'existe qu'au niveau fonctionnel. Il l'a trouvée par l'application de sa nouvelle généralisation du théorème de RICE et KAC sur la fréquence d'arrivée des zéros d'une fonction stochastique. Malheureusement, le niveau fonctionnel a bien trop de complexité pour s'appliquer tout de suite au problème de la diffusion.

Dans ma conférence, j'ai rappelé la relation Lagrangienne introduite par Sir GEOFFREY TAYLOR pour exprimer le déplacement en fonction de l'autocorrélation de vitesse en suivant la particule. Bien que le déplacement soit commandé par les tourbillons

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Je m'excuse de lire ma conférence, mais autrement vous ne pourriez pas reconnaître du tout qu'elle est vraiment une langue.

Je veux remercier les personnes qui ont aidé à la traduction, particulièrement M. Verollet.

La diffusion à partir d'une source ponctuelle dans un champ de turbulence homogène s'exprime plus aisément dans le système de coordonnées de Lagrange. Mais dans ce système les équations de Navier-Stokes sont très difficiles à utiliser. Donc le problème théorique peut se définir comme la recherche d'une relation entre les fonctions statistiques du système de Lagrange et du système d'Euler. L'analyse dynamique et l'expérience semblent toutes deux plus simples dans le système d'Euler. Notre premier objectif est de prédéterminer la valeur du carré moyen du déplacement d'une particule.

Dans la première conférence, sur « Le caractère mathématique du problème de la relation entre les fonctions statistiques Lagrangiennes et Eulériennes », le Docteur Lumley a démontré que les moyennes prises en un seul point dans chacun des deux systèmes sont égales l'une à l'autre si la densité du fluide est constante. C'est un résultat que tout le monde a appliqué intuitivement. Je pense qu'il soulève des questions sérieuses pour les problèmes dans lesquels la densité peut avoir de grandes fluctuations.

Il a traité également de l'établissement d'une relation entre les champs de vitesse Lagrangien et Eulérien. Une telle relation n'existe qu'au niveau fonctionnel. Il l'a trouvée par l'application de sa nouvelle généralisation du théorème de Rice et Kac sur la fréquence d'arrivée des zéros d'une fonction stochastique. Malheureusement, le niveau fonctionnel a bien trop de complexité pour s'appliquer tout de suite au problème de la diffusion.

Dans ma conférence, j'ai rappelé la relation Lagrangienne introduite par Sir Geoffrey Taylor pour exprimer le déplacement en fonction de l'autocorrélation de vitesse en suivant la particule. Bien que le déplacement soit commandé par les tourbillons

à basses fréquences, on peut trouver des fonctions associées qui sont commandées par les hautes fréquences obéissant aux hypothèses de KOLMOGOROV.

Cette théorie est plus directement applicable à l'analyse de la diffusion relative de deux particules, que OBUKHOV et BATCHELOR ont discutée. Ils pouvaient de cette façon déduire la loi de RICHARDSON.

Récemment LIN a attaqué les problèmes de la diffusion d'une ou de deux particules sans utiliser la théorie de KOLMOGOROV. Il arrive aux mêmes résultats, mais on ne sait si ses hypothèses sont plus ou moins restrictives.

Le Docteur SAFFMAN a démontré d'une façon convaincante que sa récente généralisation des ouvrages de TAYLOR et TOWNSEND, sur la diffusion simultanée par la turbulence et par l'agitation moléculaire, est correcte quand la durée de la diffusion est assez courte. Il a présenté, grâce à la théorie du transfert de la chaleur, les mêmes résultats qu'il a publiés il y a deux ans, obtenus avec une théorie qui tient compte à la fois des vitesses du fluide et des molécules. Il trouve un résultat surprenant : l'interaction réduit la diffusion.

Le Professeur HINZE a passé en revue les ouvrages sur la diffusion dans des écoulements turbulents avec tension de frottement. Deux cas sont relativement simples et ont été analysés : le premier est la diffusion longitudinale dans un type (par TAYLOR et par BATCHELOR, BINNIE et PHILLIPS) loin du point d'émission; l'autre cas est la diffusion à partir d'un point dans la turbulence générale lorsque l'on fait tendre le temps vers zéro.

La diffusion a beaucoup d'intérêt intrinsèque. Certaines recherches pourront être intéressantes dans ce domaine. Par exemple :

(1) Il faut faire plus d'expériences dans les souffleries à air et à eau, avec une grande étendue de nombres de Reynolds et de Prandtl.

(2) Sir GEOFFREY TAYLOR a suggéré de calculer l'effet dynamique des contaminants utilisés dans les expériences.

(3) Avec les grandes calculatrices digitales électroniques il sera peut-être possible de poursuivre le problème par la voie théorique. Par exemple, on pourra essayer la méthode « Monte-Carlo » pour intégrer des densités de probabilité fonctionnelle.

(4) Les calculatrices pourront aussi être utiles pour des expériences numériques, par exemple pour les « marches aléatoires » qui ont des caractéristiques à la fois Eulériennes et Lagrangiennes.

(5) Pour les mathématiciens, je suggère d'étudier les fonctions qui dépendent d'elles-mêmes.